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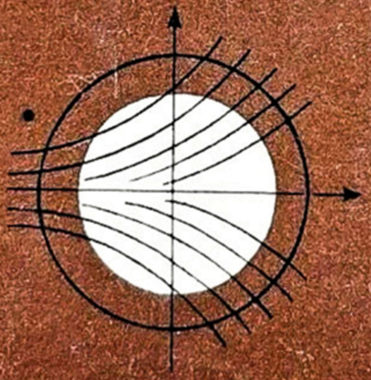
PROBLEMS AND EXERCISES IN THE CALCULUS OF VARIATIONS



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ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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PROBLEMS AND EXERCISES IN THE CALCULUS OF VARIATIONS

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CONTENTS

Introduction 6
Preliminary Remarks 7
CHAPTER I. THE EXTREMUM OF A FUNCTION OF MANY VARIABLES
1. Absolute Extremum 9
2. Conditional Extremum 19
CHAPTER II. EXTREMA OF FUNCTIONALS
3. The Functional. The Variation of a Functional and Its Properties 27
4. An Elementary Problem in the Calculus of Variations. Euler's Equation 54
5. Generalizations of the Elementary Problem of the Calculus of Variations 75
6. Invariance of Euler's Equation 85
7. Field of Extremals 88
8. Sufficient Conditions for the Extremum of a Functional 102
9. Conditional Extremum 118
10. Moving Boundary Problems 137
11. Discontinuous Problems. One-Sided Variations . . . 151
12. The Hamilton-Jacobi Theory. The Variational Principles of Mechanics 161
CHAPTER III. DIRECT METHODS IN THE CALCULUS OF VARIATIONS
13. Euler's Finite-Difference Method 177
14. Ritz Method. Kantorovich Method 179
15. Variational Methods for Finding Eigenvalues and Eigenfunctions 188
ANSWERS AND HINTS 202
BIBLIOGRAPHY 215
INDEX 217

INTRODUCTION

The modern engineer often has to do with problems that require a sound mathematical background and set skills in the use of diverse mathematical methods. Expanding the mathematical outlook of the engineer contributes appreciably to new advances in technology.

The calculus of variations is one of the most important divisions of classical mathematical analysis as regards applications.

The authors' aim was to supply the reader with a certain minimum of problems covering the basic divisions of the classical calculus of variations, and they deliberately avoided questions pertaining to the theory of optimal control.

The present text was specially designed for students using "Differential Equations and the Calculus of Variations" by L. E. Elsgolts (Mir Publishers, Moscow, 1970) and the reference book "The Calculus of Variations and Integral Equations" by L. Ya. Tslaf.

The authors

PRELIMINARY REMARKS

1. If A is an arbitrary set of elements, then the assertion "*element a belongs to (is a member of) the set A*" is symbolized as follows: $a \in A$.

The symbolism $a \notin A$ (or $a \bar{\in} A$) is used to denote that the element a does not belong to (is not a member of) the set A.

If A and B are sets, then the assertion "*A is a subset of the set B*" (symbolized thus: $A \subset B$) signifies that any element x of the set A is also a member of the set B.

2. The union and intersection of two sets A and B are defined as follows:

the *union* $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ is the collection of elements x belonging to at least one of the sets A and B;

the *intersection* $A \cap B = \{x \mid x \in A,\ x \in B\}$ is the set of elements x lying both in A and in B.

3. If A is a certain set of real numbers, then the *least upper bound* (*supremum*) of A is the smallest real number M such that $a \leqslant M$ for all $a \in A$. In other words, M is the least upper bound of A if for any $a \in A$ we have $a \leqslant M$, but no matter what $\varepsilon > 0$, even if it is arbitrarily small, there will be at least one element $b \in A$ such that $M - \varepsilon < b$.

If no such number M exists, then the supremum of A is taken to be $+\infty$.

In both cases, the supremum of the set A is denoted by $\sup A$. An analogous definition is given for the *greatest lower bound* (*infimum*) *of the set A*, which is denoted by $\inf A$.

4. A *linear space* is a set R of elements $x, y, z, \ldots$ of an arbitrary nature, for which are defined the operations of addition and scalar multiplication and for which the following axioms hold true:

(1) $x + y = y + x$;

(2) $(x + y) + z = x + (y + z)$;

(3) there exists an element 0 (zero element) such that $x + 0 = x$ for every $x \in R$;

(4) for every $x \in R$ there exists an element $-x$ (additive inverse element) such that $x + (-x) = 0$;

(5) $1 \cdot x = x$;

(6) $\alpha(\beta x) = (\alpha\beta)x$;

(7) $(\alpha + \beta)x = \alpha x + \beta x$;

(8) $\alpha (x + y) = \alpha x + \alpha y$.

5. A *linear space* R is a *normed linear space* if to every element $x \in R$ there is associated a nonnegative real number $\| x \|$ called the *norm* of that element, and:

(1) $\| x \| = 0$ only when $x = 0$;
(2) $\| \alpha x \| = | \alpha | \cdot \| x \|$;
(3) $\| x + y \| \leqslant \| x \| + \| y \|$ (triangle axiom for norms).

6. A set M of elements $x, y, z, \ldots$ of any nature whatsoever is a *metric space* if to each pair of elements x, y in M there is associated a nonnegative real number $\rho (x, y)$ such that

(1) $\rho (x, y) = 0$ if and only if $x = y$ (identity axiom);
(2) $\rho (x, y) = \rho (y, x)$ (symmetry axiom);
(3) $\rho (x, y) + \rho (y, z) \geqslant \rho (x, z)$ (triangle axiom).

The number $\rho (x, y)$ is called the *distance between the elements* x and y.

Every normed linear space is a metric space, for it suffices to put $\rho (x, y) = \| x - y \|$.

7. The *space* $C [a, b]$ is the space of all functions $y (x)$ continuous on $[a, b]$:

$$\| y \|_C = \max_{a \leqslant x \leqslant b} | y (x) |$$

The *space* $C_1 [a, b]$ is the space of all functions $y (x)$ continuous on $[a, b]$ together with their first derivatives:

$$\| y \|_{C_1} = \max_{a \leqslant x \leqslant b} | y (x) | + \max_{a \leqslant x \leqslant b} | y' (x) |$$

The *space* $C_n [a, b]$ is the space of all functions $y (x)$ continuous on $[a, b]$ together with derivatives up to order n inclusive (n is a fixed positive integer):

$$\| y \|_{C_n} = \sum_{k=0}^{n} \max_{a \leqslant x \leqslant b} | y^{(k)} (x) |$$

Sometimes the *norm of the element* $y (x)$ in $C_n [a, b]$ is defined thus:

$$\| y \| = \max_{a \leqslant x \leqslant b} \{ | y (x) |, \ | y' (x) |, \ \ldots, \ | y^{(n)} (x) | \}$$

CHAPTER I

THE EXTREMUM OF A FUNCTION OF MANY VARIABLES

1. ABSOLUTE EXTREMUM

Let a function $f(x_1, x_2, \ldots, x_n)$ or, briefly, $f(x)$, be given in a domain D of Euclidean n-dimensional space E^n.

We will say that the function $f(x)$ *attains its maximum (minimum) value* at a point $x_0 \in D$ if, no matter what the point $x \in D$, we have

$$f(x) \leqslant f(x_0) \quad (f(x) \geqslant f(x_0))$$

Weierstrass theorem. *Any function continuous in a closed bounded domain attains its maximum and minimum values in that region.*

Definition 1. Let a function $f(x)$ be defined in a domain $D \subset E^n$. The point $x^{(0)} = (x_1^0, \ldots, x_n^0) \in D$ is called the *point of strict maximum* (or *point of strict minimum*) of the function $f(x)$ if there exists a neighbourhood $\Omega(x^{(0)})$ of the point $x^{(0)}$ such that the inequality $f(x) < f(x^{(0)})$ (or $f(x) > f(x^{(0)})$) holds for all points $x \in \Omega(x^{(0)}) \cap D$, $x \neq x^{(0)}$.

Characteristic of the point of strict maximum (or point of strict minimum) is the fact that

$$\Delta f = f(x) - f(x^{(0)}) < 0 \ (\text{or } \Delta f > 0)$$

or all $x \in \Omega(x^{(0)}) \cap D$, $x \neq x^{(0)}$.

If for the point $x^{(0)}$ there exists a neighbourhood $\Omega(x^{(0)})$ such that for all points $x \in \Omega(x^{(0)}) \cap D$ the condition $f(x) \leqslant f(x^{(0)})$ (or $f(x) \geqslant f(x^{(0)})$) holds, then the point

$x^{(0)}$ is simply called the *maximum point* (or the *minimum point*).

Definition 2. The maximum point and the minimum point of a function $f(x)$ are generically termed the *extremum points* of that function.

1. Use the definition to find the extremum points of the functions:

(a) $f(x_1, x_2) = x_1^2 + x_2^2$;

(b) $f(x_1, x_2) = \begin{cases} x_1^2 + x_2^2, & x_1^2 + x_2^2 \neq 0, \\ 1, & x_1^2 + x_2^2 = 0; \end{cases}$

(c) $f(x_1, x_2) = x_1^2 - x_2^3$

in the domain $D\{x_1^2 + x_2^2 \leqslant 1\}$.

Theorem 1 (necessary condition for an extremum). *Let a function $f(x)$, $x = (x_1, x_2, \ldots, x_n)$, be defined in a neighbourhood of the point $x^{(0)} = (x_1^0, x_2^0, \ldots, x_n^0)$. If this point is an extremum point of the function $f(x)$ and if at this point there exist derivatives $\frac{\partial f}{\partial x_j}$ $(j = 1, 2, \ldots, n)$, then they are equal to zero:*

$$\frac{\partial f(x^{(0)})}{\partial x_j} = 0 \quad (j = 1, 2, \ldots, n)$$

If the function $f(x)$ is differentiable at the extremum point $x^{(0)}$, then its differential is zero at this point: $df(x^{(0)}) = 0$.

Example 1. Find the extremum points of the function $z = x^2 + y^2$.

Solution. The extremum points are among those points for which $dz = 0$. In our case, $dz = 2x\,dx + 2y\,dy$. The condition $dz = 0$ holds true in the single point $x = 0$, $y = 0$. Indeed, if $x = y = 0$, then $dz = 0$. Conversely, let $dz = 0$; taking advantage of the arbitrariness of dx and dy, choose $dy = 0$. Then $0 = dz = 2x\,dx$ and, by virtue of the arbitrariness of dx, it follows that $x = 0$. In similar fashion, we find that $y = 0$. At the point (0, 0) we have $z = 0$, at all other points, $z = x^2 + y^2 > 0$. Therefore, the point (0, 0) is the strict minimum point for the function $z = x^2 + y^2$.

If, by i ıding functions that are not differentiable at separate p ts, we extend the class of functions in which an extrem is sought, we arrive at the following necessary condition an extremum.

If $x^{(0)}$ is *xtremum point of the function $f(x_1, x_2, \ldots, x_n)$,*

then at this *nt each partial derivative $\frac{\partial f}{\partial x_i}$ $(i = 1, 2, \ldots, n)$*

is either ze *r does not exist.*

Exampl We consider the upper nappe of the cone $z^2 = x^2 +$ $z \geqslant 0$. It is obvious that the function z has a

the point $O(0, 0)$. But $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ do not

ıaıst at 1 ɔoint.

Definiti . Points at which the necessary condition for an ex ım of a function $f(x)$ is fulfilled are called the *critic* *nts of the function.*

The po $x^{(0)}$ at which $df(x^{(0)}) = 0$ are called *stationary points of* ı *nction* $f(x)$.

The cor on $df(x^{(0)}) = 0$ is equivalent to the condition

$$\frac{\partial f(x^{(0)})}{\partial x_i} = 0 \quad (i = 1, 2, \ldots, n)$$

The existence of a critical point does not guarantee the existence of an extremum of the function. For example, with respect to the function $z = x^2 - y^2$ the point (0, 0) is a stationary point, but there is no extremum of the function z at that point: in any arbitrarily small neighbourhood of the point (0, 0) the function assumes both positive and negative values.

1°. Sufficient conditions for a strict extremum.

Definition 4. The quadratic form

$$A(x) = A(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

$$a_{ij} = a_{ji};\ i, j = 1, 2, \ldots, n$$

is *positive definite* (or *negative definite*) if $A(x) > 0$ (or $A(x) < 0$) for any point $x \in E^n$, $x \neq 0$, and vanishes only when $x = 0$, that is, when $x_1 = x_2 = \ldots = x_n = 0$.

A quadratic form is *nonnegative* if it never assumes negative values. For example, the forms $x_1^2 + x_2^2 + \ldots + x_n^2$

and $(x_1 + x_2 + \ldots + x_n)^2$ are nonnegative forms. The former is positive definite since it vanishes only when $x_1 = x_2 = \ldots = x_n = 0$, the latter form will not be positive definite since it vanishes, for example, when $x_1 = 1$, $x_2 = -1$, $x_3 = x_4 = \ldots = x_n = 0$.

A quadratic form which is positive or negative definite is called a *definite quadratic form.*

A quadratic form that takes on both positive and negative values is called an *indefinite quadratic form.*

Theorem 2 (sufficient conditions for a strict extremum). *Let a function $f(x)$ be defined and have continuous second derivatives in some neighbourhood of a point $x^{(0)} = (x_1^0, x_2^0, \ldots, x_n^0)$ and let $x^{(0)}$ be a stationary point of $f(x)$. If the quadratic form*

$$A(dx_1, dx_2, \ldots, dx_n) = \sum_{i,\, j=1}^{n} \frac{\partial^2 f(x^{(0)})}{\partial x_i \partial x_j} dx_i dx_j \qquad (1)$$

that is, the second differential of the function f at the point $x^{(0)}$ is a positive definite (negative definite) quadratic form, then the point $x^{(0)}$ is a strict minimum point (strict maximum point); if the quadratic form (1) is indefinite, then there is no extremum at the point $x^{(0)}$.

Sylvester's criterion for positive definiteness of a quadratic form. For the quadratic form

$$A(x) = A(x_1, x_2, \ldots, x_n) = \sum_{i,\, j=1}^{n} a_{ij} x_i x_j \qquad (2)$$

in which $a_{ij} = a_{ji}$, $i, j = 1, 2, \ldots, n$, to be positive definite, it is necessary and sufficient that

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$$

$$\ldots, \quad \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} > 0$$

For the quadratic form (2) to be negative definite, it is necessary and sufficient that

$$a_{11}<0,\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0,\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}<0$$

$$\ldots,\ \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}(-1)^n>0$$

The case of $n = 2$. Let the function $f(x, y)$ be defined and let it have continuous second partial derivatives in some neighbourhood of the point (x_0, y_0) and let (x_0, y_0) be a stationary point; that is,

$$f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0 \tag{3}$$

Now, if at the point (x_0, y_0)

$$f''_{xx}f''_{yy} - (f''_{xy})^2 > 0 \tag{4}$$

then it is an extremum point, namely a maximum if at that point

$$f''_{xx}<0 \quad (f''_{yy}<0)$$

and a minimum if

$$f''_{xx}>0 \quad (f''_{yy}>0)$$

But if at the point (x_0, y_0)

$$f''_{xx}f''_{yy} - (f''_{xy})^2 < 0$$

then there is no extremum at the point (x_0, y_0). Finally, when

$$f''_{xx}f''_{yy} - (f''_{xy})^2 = 0$$

at the point (x_0, y_0), there may or may not be an extremum at that point. In the latter case, a supplementary investigation is required.

Example 3. Consider the functions $z = x^4 + y^4$, $z = -x^4 - y^4$, $z = x^4 - y^4$. The point (0, 0) is a stationary

point for each of these functions and for each of them $z''_{xx} \cdot z''_{yy} - (z''_{xy})^2 = 0$ at that point.

It is readily seen that the point (0, 0) is a minimum point for the first function, a maximum point with respect to the second function, and is not an extremum point relative to the third function. Indeed, in all three cases we have $z\,(0,\ 0) = 0$, but in the first case the values of the function are positive in any neighbourhood of the point (0, 0), except the point itself, in the second case they are negative, and in the third case the function $z = x^4 - y^4$ takes on both positive values (for instance, $y = 0$ when $x \neq 0$) and negative values (for example, $y \neq 0$ when $x = 0$) arbitrarily close to the coordinate origin.

Example 4. Find the extremum of the following function of three variables:

$$f = x^2 + y^2 + z^2 - xy + x - 2z$$

Solution. Find the stationary points of the given function f. To do this, set up the system of equations

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x - y + 1 = 0, \\ \frac{\partial f}{\partial y} &= 2y - x = 0, \\ \frac{\partial f}{\partial z} &= 2z - 2 = 0 \end{aligned}\right\}$$

Solving this system, we get $x_0 = -\frac{2}{3}$, $y_0 = -\frac{1}{3}$, $z_0 = 1$.

Set up the quadratic form (1) at the point $P_0\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

We have

$$\begin{aligned} &f''_{xx} = 2, && f''_{xy} = -1, && f''_{xz} = 0, \\ &f''_{yx} = -1, && f''_{yy} = 2, && f''_{yz} = 0, \\ &f''_{zx} = 0, && f''_{zy} = 0, && f''_{zz} = 2 \end{aligned}$$

At the point P_0 we obtain

$$\begin{aligned} &a_{11} = 2, && a_{12} = -1, && a_{13} = 0, \\ &a_{21} = -1, && a_{22} = 2, && a_{23} = 0, \\ &a_{31} = 0, && a_{32} = 0, && a_{33} = 2 \end{aligned}$$

so that

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0,$$

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 > 0$$

Taking advantage of Sylvester's criterion, we conclude that the quadratic form is positive definite and, hence, by Theorem 2, the point P_0 is a strict minimum point, and $f(P_0) = -\frac{4}{3}$.

Example 5. Find the extremum of the following function of two variables:

$$z = x^3y^2 (6 - x - y)$$

Solution. We find the stationary points:

$$\left.\begin{aligned} z'_x &= 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0, \\ z'_y &= 12x^3y - 2x^4y - 3x^3y^2 = 0 \end{aligned}\right\}$$

whence $x_1 = 0$, $y_1 = 0$, and $x_2 = 3$, $y_2 = 2$. We obtain two stationary points, P_1 (0, 0) and P_2 (3, 2).

Let us find the second derivatives of the given function:

$$\begin{aligned} z''_{xx} &= 36xy^2 - 12x^2y^2 - 6xy^3, \\ z''_{yy} &= 12x^3 - 2x^4 - 6x^3y, \\ z''_{xy} &= 36x^2y - 8x^3y - 9x^2y^2 \end{aligned}$$

At the point P_1 we have $z''_{xx} = z''_{yy} = z''_{xy} = 0$ so that $z''_{xx} \cdot z''_{yy} - (z''_{xy})^2 = 0$ and the question of the existence of an extremum at this point remains open. To resolve this question we have to invoke higher derivatives.

At the point P_2 we have $z''_{xx} = -144$, $z''_{yy} = -162$, $z''_{xy} = -108$. Obviously, $z''_{xx} \cdot z''_{yy} - (z''_{xy})^2 > 0$ and since $z''_{xx} < 0$ it follows that at the point P_2 (3, 2) there is a maximum, and $z_{\max} = 108$.

Test the following functions for maxima and minima.

2. $f = (x - 1)^2 - 2y^2$.

3. $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

4. $f = (x^2 + y^2)\, e^{-(x^2+y^2)}$.

5. $f = \dfrac{1+x-y}{\sqrt{1+x^2+y^2}}$.

6. $f = x + \dfrac{y^2}{4x} + \dfrac{z^2}{y} + \dfrac{2}{z}$ $(x > 0,\ y > 0,\ z > 0)$.

7. $f = x^2 - xy + y^2 - 2x + y$.

8. $f = \sin x \cdot \sin y \cdot \sin (x + y)$ $(0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi)$.

9. $f = x_1 \cdot x_2^2 \ldots x_n^n\, (1 - x_1 - 2x_2 - \ldots - nx_n)$
$(x_1 > 0,\ x_2 > 0,\ \ldots,\ x_n > 0)$.

10. Show that the function $z = (1 + e^y) \cos x - ye^y$ has an infinitude of maxima and not a single minimum.

11. State whether the following condition is sufficient for the minimum of the function $f(x, y)$ at the point $M_0(x_0, y_0)$: that this function have a minimum along every straight line passing through the point M_0. Consider the example $f(x, y) = (x - y^2)(2x - y^2)$.

12. Show that, unlike a function of one variable, the existence in a domain D of a unique extremum (maximum or minimum) for a function of two variables does not yet signify that this extremum necessarily yields the maximum or minimum value of the function throughout the domain:

(a) $z = x^2 - y^2 + 2e^{-x^2}$,
$-\infty < x < +\infty, \quad -\infty < y < +\infty$;

(b) $z = x^3 - 4x^2 + 2xy - y^2$,
$D\ \{-5 \leqslant x \leqslant 5, \quad -1 \leqslant y \leqslant 1\}$.

13. Let there be given a periodic function $f(x)$ with period 2π. By means of selecting coefficients α_k, β_k, it is required to find, from among all trigonometric polynomials of order n, that polynomial

$$\frac{\alpha_0}{2} + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx)$$

for which the root-mean-square deviation defined by the equation

$$\delta_n^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(x) - \frac{\alpha_0}{2} - \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx) \right]^2 dx$$

has a minimum value.

2°. **The method of steepest descent.** Suppose the problem is to find the minimum of a function $f(x)$, where $x = (x_1, x_2, \ldots, x_m)$. Take a certain point $x^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ and compute the gradient of the function $f(x)$ at that point:

$$\operatorname{grad} f(x^0) = \sum_{i=1}^{m} \frac{\partial f(x^0)}{\partial x_i} e_i$$

where $e_1, e_2, \ldots, e_n$ constitute an orthonormal basis in the space R^n.

If $\operatorname{grad} f(x^0) \neq 0$, then we put

$$x_k^1 = x_k^0 - h_1 (\operatorname{grad} f(x^0), e_k) \quad (k = 1, 2, \ldots, m)$$

where $h_1 > 0$ is sufficiently small. If $\operatorname{grad} f(x^1) \neq 0$, then we put

$$x_k^2 = x_k^1 - h_2 (\operatorname{grad} f(x^1), e_k) \qquad (h_2 > 0)$$

and, generally, if $\operatorname{grad} f(x^{n-1}) \neq 0$, then

$$x_k^n = x_k^{n-1} - h_n (\operatorname{grad} f(x^{n-1}), e_k) \quad (k = 1, 2, \ldots, m) \qquad (h_n > 0)$$

Under definite conditions (see [19]) we get a monotonic decreasing sequence $\{f(x^n)\}$. If $x^n \to \tilde{x}$ and $\tilde{x}$ is a minimum point of the function $f(x)$, then $\operatorname{grad} f(x^n) \to 0$ as $n \to \infty$.

Example 6. Find the minimum point of the function $f(x) = x^2$.

Solution. Let us take the point $x^0 = 1$. We have

$$\operatorname{grad} f(x^0) = 2x^0 i = 2i \neq 0$$

Therefore

$$x^1 = x^0 - h \cdot 2 = 1 - 2h, \qquad \text{where } h > 0$$

Furthermore

$$\operatorname{grad} f(x^1) = 2(1 - 2h)\, i$$

If $h \neq \frac{1}{2}$, then $\operatorname{grad} f(x^1) \neq 0$ and

$$x^2 = x^1 - 2h(1 - 2h) = (1 - 2h)^2$$

Continuing this process, we find

$$x^n = (1 - 2h)^n$$

Clearly, if $0 < h < 1$, then $x^n \to 0$ as $n \to \infty$. The point $x = 0$ is the minimum point of the function $f(x) = x^2$. But if $h = 1/2$, then $x^1 = 0$, $\operatorname{grad} f(x^1) = 0$ and we obtain the stationary sequence $\{0\}$ whose limit is zero.

Example 7. Find the minimum point of the function $f(x, y) = x^2 + y^2$.

Solution. Let us, for example, take the point (1, 1), that is, $x^0 = 1$, $y^0 = 1$. We find

$$\operatorname{grad} f(1, 1) = 2i + 2j$$

Since $\operatorname{grad} f(1, 1) \neq 0$, we put

$$\left.\begin{aligned} x^1 &= x^0 - 2x^0 h = 1 - 2h, \\ y^1 &= y^0 - 2y^0 h = 1 - 2h \end{aligned}\right\} \quad (h > 0)$$

We have

$$\operatorname{grad} f(x^1, y^1) = 2(1-2h)\, i + 2(1-2h)\, j \neq 0 \quad \left(h \neq \frac{1}{2}\right)$$

and so we take

$$\left.\begin{aligned} x^2 &= x^1 - 2x^1 \cdot h = (1-2h)^2, \\ y^2 &= y^1 - 2y^1 \cdot h = (1-2h)^2 \end{aligned}\right\} \quad \left(h > 0,\ h \neq \frac{1}{2}\right)$$

Continuing this process, we obtain

$$\begin{aligned} x^n &= (1 - 2h)^n, \\ y^n &= (1 - 2h)^n \end{aligned}$$

so that when $0 < h < 1$ we will have a sequence of points $M_n(x^n, y^n)$ that converges to the minimum point $M(0, 0)$ of the given function. It is obvious that

$$\operatorname{grad} f(x^n, y^n) = 2(1 - 2h)^n i + 2(1 - 2h)^n j \to 0$$

as $n \to \infty$

Thus, the minimum point of the function $f(x, y) = x^2 + y^2$ is the point (0, 0).

Using the method of steepest descent, find the minimum point of the function

$$z = x^2 + y^2 - 2x + 4y + 5$$

2. CONDITIONAL EXTREMUM

Suppose we have a function $z = f(x_1, x_2, \ldots, x_n)$ of n variables defined in some domain D of the space E^n.

Besides, let there be imposed on $x_1, x_2, \ldots, x_n$ m accessory conditions ($m < n$):

$$\left.\begin{aligned} &\varphi_1(x_1, x_2, \ldots, x_n) = 0, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\varphi_m(x_1, x_2, \ldots, x_n) = 0 \end{aligned}\right\} \qquad (1)$$

which are called the equations of constraint.

Let $x^{(0)} = (x_1^0, x_2^0, \ldots, x_n^0)$ be an interior point of the domain D.

We say that at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ the function $f(x_1, x_2, \ldots, x_n)$ has a *conditional maximum* (or *conditional minimum*) if the inequality

$$f(x_1, x_2, \ldots, x_n) \leqslant f(x_1^0, x_2^0, \ldots, x_n^0) \qquad (2)$$

(or $f(x_1, x_2, \ldots, x_n) \geqslant f(x_1^0, x_2^0, \ldots, x_n^0)$) holds true in some neighbourhood of the point $(x_1^0, x_2^0, \ldots, x_n^0)$ provided that the points $(x_1, x_2, \ldots, x_n)$ and $(x_1^0, x_2^0, \ldots, x_n^0)$ satisfy the constraint equations (1).

Example 1. The function $z = x^2 + y^2$ has an absolute minimum equal to zero at the point (0, 0). We adjoin the constraint equation $x + y - 1 = 0$, that is, we seek the minimum of the z-coordinates of the points of the surface $z = x^2 + y^2$ solely for those values of x and y that satisfy the equation $x + y - 1 = 0$. A conditional minimum cannot be attained at the point (0, 0) because this point does not satisfy the constraint equation. Let us solve the constraint equation $x + y - 1 = 0$ for y and substitute the value found, $y = 1 - x$, into the equation of the surface. This yields $z = x^2 + (1 - x)^2$, which is a function of one

variable. Investigating it for an extremum, we find $x_{cr} = \frac{1}{2}$, $z_{\min} = \frac{1}{2}$. By virtue of the constraint equation, find $y_{cr} = \frac{1}{2}$. The point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the vertex of a parabola obtained in the intersection of the paraboloid $z = x^2 + y^2$ cut by the plane $x + y - 1 = 0$.

We can reason analogously in the more general case as well.

Suppose we are seeking the conditional extremum of the function $z = f(x, y)$ given the constraint $\varphi(x, y) = 0$. Also suppose that for the given values of x and y the equation $\varphi(x, y) = 0$ defines y as a single-valued differentiable function $y = \psi(x)$. Substituting the function $\psi(x)$ in place of y into the function $f(x, y)$, we get a function of the single variable x: $z = f(x, \psi(x)) = F(x)$. The absolute extremum of the function $F(x)$ is the desired conditional extremum of the function $f(x, y)$, given the constraint $\varphi(x, y) = 0$. This method is not always convenient in a practical sense since it requires the actual solution of the equation $\varphi(x, y) = 0$ for some variable.

In seeking the extremal values of the function $z = f(x_1, x_2, \ldots, x_n)$ in the case of constraints (1), use is made of the Lagrangian method of undetermined multipliers.

The Lagrangian multiplier method. Suppose that

(1) the functions $f(x_1, x_2, \ldots, x_n)$ and $\varphi_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ have continuous partial derivatives of the first order in the domain D;

(2) $m < n$ and the rank of the matrix $\left\| \frac{\partial \varphi_i}{\partial x_j} \right\|$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, is equal to m at every point of D.

A new function (the Lagrangian function) is formed:

$$\Phi = f + \sum_{i=1}^{m} \lambda_i \varphi_i \tag{3}$$

where λ_i are undetermined constant multipliers.

The function $\Phi(x_1, x_2, \ldots, x_n)$ is investigated for an absolute extremum; that is the following system of equa-

tions is set up:

$$\frac{\partial \Phi}{\partial x_1} = 0, \quad \frac{\partial \Phi}{\partial x_2} = 0, \quad \ldots, \quad \frac{\partial \Phi}{\partial x_n} = 0 \tag{4}$$

from which and from the m equations of constraint

$$\varphi_1 = 0, \quad \varphi_2 = 0, \quad \ldots, \quad \varphi_m = 0$$

are determined the values of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ and the coordinates $(x_1, x_2, \ldots, x_n)$ of possible extremum points.

The conditions (4) are necessary conditions for an extremum both of the Lagrangian function and the original function $z = f(x_1, x_2, \ldots, x_n)$.

If the point $(x_1^0, x_2^0, \ldots, x_n^0)$ is a conditional extremum point of the function $f(x_1, x_2, \ldots, x_n)$, then it is a stationary point of the Lagrangian function; that is, at this point $\frac{\partial \Phi}{\partial x_i} = 0$ $(i = 1, 2, \ldots, n)$. In order to investigate the stationary point $(x_1^0, x_2^0, \ldots, x_n^0)$ of the Lagrangian function $\Phi(x_1, x_2, \ldots, x_n)$ for a conditional extremum, it is necessary to set up the quadratic form

$$B(dx_1, dx_2, \ldots, dx_{n-m}) = \sum_{i,\, j=1}^{n-m} b_{ij}\, dx_i\, dx_j \tag{5}$$

that is, the second differential of the Lagrangian function at this point with allowance made for the conditions

$$\frac{\partial \varphi_i}{\partial x_1} dx_1 + \frac{\partial \varphi_i}{\partial x_2} dx_2 + \ldots + \frac{\partial \varphi_i}{\partial x_n} dx_n = 0 \quad (i = 1, 2, \ldots, m) \tag{6}$$

If the quadratic form (5) is definite, then at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ there will be a strict conditional extremum, namely: a strict conditional maximum if the quadratic form (5) is negative definite, and a strict conditional minimum if the quadratic form (5) is positive definite.

But if the quadratic form (5) is indefinite, then the point $(x_1^0, x_2^0, \ldots, x_n^0)$ is not a conditional extremum point.

Thus, the existence at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ of an absolute maximum (minimum) of the Lagrangian function (for the values found: $\lambda_1, \lambda_2, \ldots, \lambda_m$) implies the

existence at that point of a conditional maximum (minimum) of the function $z = f(x_1, x_2, \ldots, x_n)$, given the constraints

$$\varphi_i(x_1, x_2, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, m)$$

The absence of an absolute extremum of the Lagrangian function $\Phi(x_1, x_2, \ldots, x_n)$ does not yet signify the absence of a conditional extremum of the function $f(x_1, x_2, \ldots, x_n)$.

Example 2. Find the extremum of the function $z = xy$ provided $y - x = 0$.

Solution. Form the Lagrangian function

$$\Phi(x, y) = xy + \lambda(y - x)$$

and write out the appropriate system for determining λ and the coordinates of the possible extremum points:

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial x} &= y - \lambda = 0, \\ \frac{\partial \Phi}{\partial y} &= x + \lambda = 0, \\ y - x &= 0 \end{aligned}\right\} \tag{7}$$

From the first equation we find $\lambda = y$. Substituting into the second, we get $x + y = 0$. Thus,

$$\left.\begin{aligned} x + y &= 0, \\ y - x &= 0 \end{aligned}\right\}$$

whence $x = y = 0$. We then have $\lambda = 0$. Thus, the appropriate Lagrangian function is of the form $\Phi(x, y) = xy$. The function $\Phi(x, y)$ does not have an absolute extremum at the point (0, 0), but there is a conditional extremum of the function $z = xy$ provided that $y = x$. Indeed, in that case we have $z = x^2$, whence it is seen that there is a conditional minimum at the point (0, 0).

Example 3. Find the conditional extremum of the function

$$f(x, y, z) = xyz \tag{8}$$

subject to the conditions

$$\left.\begin{aligned} \varphi_1(x, y, z) &= x + y - z - 3 = 0, \\ \varphi_2(x, y, z) &= x - y - z - 8 = 0 \end{aligned}\right\} \tag{9}$$

Solution. Form the Lagrangian function

$$\Phi(x, y, z) = xyz + \lambda_1 (x + y - z - 3) + \lambda_2 (x - y - z - 8)$$

and write out the system of equations for determining the parameters λ_1, λ_2 and the coordinates of the possible extremum points:

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial x} &= yz + \lambda_1 + \lambda_2 = 0, \\ \frac{\partial \Phi}{\partial y} &= xz + \lambda_1 - \lambda_2 = 0, \\ \frac{\partial \Phi}{\partial z} &= xy - \lambda_1 - \lambda_2 = 0, \\ x + y - z - 3 &= 0, \\ x - y - z - 8 &= 0 \end{aligned}\right\} \tag{10}$$

Solving the system of equations (10), we obtain

$$\lambda_1 = \frac{11}{32}, \quad \lambda_2 = -\frac{231}{32}, \quad x = \frac{11}{4}, \quad y = -\frac{5}{2}, \quad z = -\frac{11}{4}$$

The second differential of the function $\Phi(x, y, z)$ is

$$d^2\Phi = \frac{\partial^2 \Phi}{\partial x^2} dx^2 + \frac{\partial^2 \Phi}{\partial y^2} dy^2 + \frac{\partial^2 \Phi}{\partial z^2} dz^2 + 2 \frac{\partial^2 \Phi}{\partial x \, \partial y} dx \, dy + 2 \frac{\partial^2 \Phi}{\partial x \, \partial z} dx \, dz + 2 \frac{\partial^2 \Phi}{\partial y \, \partial z} dy \, dz$$

In our case

$$d^2\Phi = 2z \, dx \, dy + 2y \, dx \, dz + 2x \, dy \, dz \tag{11}$$

Taking advantage of the constraint conditions (9), we get

$$\left.\begin{aligned} dx + dy - dz &= 0, \\ dx - dy - dz &= 0 \end{aligned}\right\}$$

whence $dx = dz$, $dy = 0$. Substituting this into (11), we obtain

$$B(dx) = 2y \, dx^2$$

At the stationary point $B = -5\,dx^2 < 0$, that is, at the point $\left(\frac{11}{4}, -\frac{5}{2}, -\frac{11}{4}\right)$ we have a maximum equal to $f_{\max} = \frac{605}{32}$.

Example 4. Find the extremum of the function $z = \cos^2 x + \cos^2 y$ subject to the condition

$$y - x = \frac{\pi}{4}$$

Solution. We form the Lagrangian function

$$\Phi(x, y) = \cos^2 x + \cos^2 y + \lambda\left(y - x - \frac{\pi}{4}\right)$$

and write out the system of equations for determining the parameter λ and the coordinates of the possible extremum points:

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial x} &= -2\cos x \sin x - \lambda = 0, \\ \frac{\partial \Phi}{\partial y} &= -2\cos y \sin y + \lambda = 0, \\ y - x - \frac{\pi}{4} &= 0 \end{aligned}\right\}$$

or

$$\sin 2x = -\lambda, \tag{12}$$

$$\sin 2y = \lambda, \tag{13}$$

$$y - x = \frac{\pi}{4} \tag{14}$$

From (12) and (13) we have $\sin 2x + \sin 2y = 0$ or

$$2 \sin (x + y) \cos (y - x) = 0 \tag{15}$$

According to (14) we have $\cos (y - x) = \frac{\sqrt{2}}{2} \neq 0$, and therefore from (15) we find that $\sin (x + y) = 0$, whence

$$x + y = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots \tag{16}$$

Solving equations (14) and (16) simultaneously, we get

$$x = \frac{k\pi}{2} - \frac{\pi}{8}, \quad y = \frac{k\pi}{2} + \frac{\pi}{8}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{17}$$

Let us find the second derivatives of the function $\Phi(x, y)$:

$$\frac{\partial^2\Phi}{\partial x^2} = -2\cos 2x, \quad \frac{\partial^2\Phi}{\partial x\,\partial y} = 0, \quad \frac{\partial^2\Phi}{\partial y^2} = -2\cos 2y$$

At the points $P_k\left(\frac{k\pi}{2} - \frac{\pi}{8}, \frac{k\pi}{2} + \frac{\pi}{8}\right)$ we have

$$\Phi''_{xx}\cdot\Phi''_{yy} - (\Phi''_{xy})^2 = 4\cos\left(k\pi - \frac{\pi}{4}\right)\cos\left(k\pi + \frac{\pi}{4}\right)$$
$$= 2\cos 2k\pi = 2 > 0$$

This means that there is a conditional extremum at the points P_k. Furthermore, when $k = 2n$,

$$\left.\frac{\partial^2\Phi}{\partial x^2}\right|_{P_{2n}} = -\sqrt{2} < 0$$

and for this reason at the points P_{2n} we have the conditional maximum

$$z_{\max} = 1 + \frac{\sqrt{2}}{2}$$

When $k = 2n+1$ we have

$$\left.\frac{\partial^2\Phi}{\partial x^2}\right|_{P_{2n+1}} = \sqrt{2} > 0$$

that is, the conditional minimum

$$z_{\min} = 1 - \frac{\sqrt{2}}{2}$$

at the points P_{2n+1}.

Find the conditional extremum in the following problems.

14. $f = xy$ for $x^2 + y^2 = 1$.

15. $f = x^2 + y^2$ for $\frac{x}{2} + \frac{y}{3} = 1$.

16. $f = xyz$ subject to the conditions $x + y + z = 5$, $xy + yz + zx = 8$.

17. $f = e^{xy}$ when $x + y = a$.

18. $f = 6 - 4x - 3y$ when $x^2 + y^2 = 1$.

19. $f = x - 2y + 2z$ when $x^2 + y^2 + z^2 = 9$.

20. $f = \sin x \sin y \sin z$ when $x + y + z = \frac{\pi}{2}$,

$$x > 0, \quad y > 0, \quad z > 0.$$

21. Prove the inequality

$$\frac{x^n+y^n}{2} \geqslant \left(\frac{x+y}{2}\right)^n, \qquad n \geqslant 1,\ x \geqslant 0,\ y \geqslant 0.$$

22. Find the maximum value of the product $xyzt$ of nonnegative numbers x, y, z, t, provided that their sum preserves the constant magnitude $x + y + z + t = 4c$.

23. Find the shortest distance from the point M (1, 0) to the ellipse $4x^2 + 9y^2 = 36$.

24. Find the distance between the parabola $y = x^2$ and the straight line $x - y = 5$.

25. Find the sides of a rectangle of maximum area inscribed in the circle $x^2 + y^2 = R^2$.

26. In a sphere of radius R inscribe a cylinder having maximum total surface area.

CHAPTER II

EXTREMA OF FUNCTIONALS

3. THE FUNCTIONAL. THE VARIATION OF A FUNCTIONAL AND ITS PROPERTIES

1°. Definitions of a functional. Proximity of curves. Suppose we have a certain class M of functions $y(x)$. If to each function $y(x) \in M$ there is associated, by some law, a definite number J, then we say that a *functional* J is defined in the class M and we write $J = J[y(x)]$.

The class M of functions $y(x)$ on which the functional $J[y(x)]$ is defined is called the *domain of definition of the functional.*

Example 1. Let $M = C[0, 1]$ be the collection of all continuous functions $y(x)$ specified on the interval $[0, 1]$ and let

$$J[y(x)] = \int_0^1 y(x)\,dx \tag{1}$$

Then $J[y(x)]$ is a functional of $y(x)$: to every function $y(x) \in C[0, 1]$ there is associated a definite value of $J[y]$. By substituting concrete functions into (1) in place of $y(x)$, we get appropriate values of $J[y]$. Thus, if $y(x) = 1$, then

$$J[1] = \int_0^1 1 \cdot dx = 1$$

If $y(x) = e^x$, then

$$J[e^x] = \int_0^1 e^x\,dx = e - 1$$

If $y(x) = \cos \pi x$, then

$$J[\cos \pi x] = \int_0^1 \cos \pi x\,dx = 0$$

Example 2. Let $M = C_1[a, b]$ be the class of functions $y(x)$ that have continuous derivatives on the interval $[a, b]$ and let

$$J[y(x)] = y'(x_0), \quad \text{where } x_0 \in [a, b] \tag{2}$$

It is clear that $J[y(x)]$ is a functional defined in the indicated class of functions for the simple reason that to each function of this class there is associated a definite number—the value of the derivative of the function at the fixed point x_0.

If, for example, $a = 1$, $b = 3$ and $x_0 = 2$, then for $y(x) = x^2$ we have

$$J[x^2] = 2x|_{x=2} = 4$$

For $y(x) = x^2 + 1$ we get $J[x^2 + 1] = 4$. For $y(x) = \ln(1 + x)$ we have $J[\ln(1 + x)] = \frac{1}{1+x}\Big|_{x=2} = \frac{1}{3}$.

Example 3. Let $M = C[-1, 1]$ be the class of functions $y(x)$ continuous on the interval $[-1, 1]$ and let $\varphi(x, y)$ be a given function defined and continuous for all $-1 \leqslant x \leqslant 1$ and for all real y. Then

$$J[y(x)] = \int_{-1}^1 \varphi[x, y(x)]\,dx \tag{3}$$

will be a functional defined on the indicated class of functions. For example, if $\varphi(x, y) = \frac{x}{1+y^2}$, then for $y(x) = x$ we have $J[x] = \int_{-1}^1 \frac{x\,dx}{1+x^2} = 0$ and for $y(x) = 1 + x$

we have

$$J[1+x]=\int_{-1}^{1}\frac{x\,dx}{1+(1+x)^2}=\ln\sqrt{5}-\arctan 2$$

Example 4. Let $M=C_1[a, b]$ be the class of functions $y(x)$ having a continuous derivative $y'(x)$ on the interval $[a, b]$. Then

$$J[y(x)]=\int_a^b\sqrt{1+y'^2(x)}\,dx \tag{4}$$

is a functional defined on that class of functions. The functional (4) geometrically describes the arc length of the curve $y=y(x)$ with ends at the points $A(a, y(a))$ and $B(b, y(b))$.

The *variation*, or *increment*, δy of the argument $y(x)$ of the functional $J[y(x)]$ is the difference between two functions $y(x)$ and $y_0(x)$ belonging to the chosen class M of functions:

$$\delta y=y(x)-y_0(x) \tag{5}$$

For the class of k-times-differentiable functions we have

$$(\delta y)^{(k)}=\delta y^{(k)}(x) \tag{6}$$

We say that the curves $y=y(x)$ and $y=y_1(x)$ specified on the interval $[a, b]$ are *close in the sense of proximity of the zeroth order* if $|y(x)-y_1(x)|$ is small on $[a, b]$. Geometrically, this means that those curves on the interval $[a, b]$ are *close as regards the ordinates.*

We will say that the curves $y=y(x)$ and $y=y_1(x)$, which are specified on the interval $[a, b]$, are close in the sense of proximity of the first order if $|y(x)-y_1(x)|$ and $|y'(x)-y_1'(x)|$ are small on $[a, b]$. Geometrically, this means that the curves on the interval $[a, b]$ are close both as to ordinates and as to directions of the tangents at the appropriate points.

The curves $y=y(x)$ and $y=y_1(x)$ are *close in the sense of proximity of the kth order* if the moduli

$$|y(x)-y_1(x)|,\ |y'(x)-y_1'(x)|,\ \ldots,\ |y^{(k)}(x)-y_1^{(k)}(x)|$$

are small on $[a, b]$.

If two curves are close in the sense of proximity of the kth order, they are all the more so close in the sense of proximity of any smaller order.

Example 5. The curves $y(x) = \frac{\sin n^2 x}{n}$, where n is sufficiently large, and $y_1(x) \equiv 0$ on $[0, \pi]$ are close in the sense of proximity of the zeroth order since the modulus of the difference

$$|y(x) - y_1(x)| = \left|\frac{\sin n^2 x}{n}\right| \leqslant \frac{1}{n}$$

That is to say, on the entire interval $[0, \pi]$ this difference is small in modulus if n is sufficiently large.

There is no proximity of the first order since

$$|y'(x) - y_1'(x)| = n|\cos n^2 x|$$

and, for example, at the points $x = \frac{2\pi}{n^2}$ we have $|y'(x) - y_1'(x)| = n$ and, hence, $|y'(x) - y_1'(x)|$ can be made arbitrarily large for sufficiently large n.

Example 6. The curves $y(x) = \frac{\sin nx}{n^2}$, where n is sufficiently great, and $y_1(x) \equiv 0$ on $[0, \pi]$ are close in the sense of proximity of the first order because

$$|y(x) - y_1(x)| = \left|\frac{\sin nx}{n^2}\right| \leqslant \frac{1}{n^2}$$

and

$$|y'(x) - y_1'(x)| = \left|\frac{\cos nx}{n}\right| \leqslant \frac{1}{n}$$

are small.

State the order of proximity of the curves in the following examples.

27. $y(x) = \frac{\cos nx}{n^2+1}$, $y_1(x) \equiv 0$ on $[0, 2\pi]$.

28. $y(x) = \frac{\sin x}{n}$, $y_1(x) \equiv 0$ on $[0, \pi]$.

29. $y(x) = \sin\frac{x}{n}$, $y_1(x) \equiv 0$ on $[0, 1]$.

The *distance between the curves* $y = f(x)$ and $y = f_1(x)$ $(a \leqslant x \leqslant b)$, where $f(x)$ and $f_1(x)$ are functions continuous on $[a, b]$, is a nonnegative number ρ equal to the maximum

of $|f_1(x) - f(x)|$ on the interval $a \leqslant x \leqslant b$:

$$\rho = \rho[f_1(x), f(x)] = \max_{a \leqslant x \leqslant b} |f_1(x) - f(x)| \qquad (7)$$

Example 7. Find the distance ρ between the curves $y = x$ and $y = x^2$ on the interval $[0, 1]$ (Fig. 1).

Solution. By definition, $\rho = \max_{0 \leqslant x \leqslant 1} |x^2 - x|$ or $\rho = \max_{0 \leqslant x \leqslant 1} (x - x^2)$. The function $y = x - x^2$ vanishes at the endpoints of the interval $[0, 1]$. Let us find the maximum of the function $y = x - x^2$ on the interval $[0, 1]$. We have

$$y' = 1 - 2x,\ y' = 0 \text{ when } x = \frac{1}{2}$$

and so

$$\rho = \max_{0 \leqslant x \leqslant 1} y = (x - x^2)\Big|_{x=\frac{1}{2}} = \frac{1}{4}$$

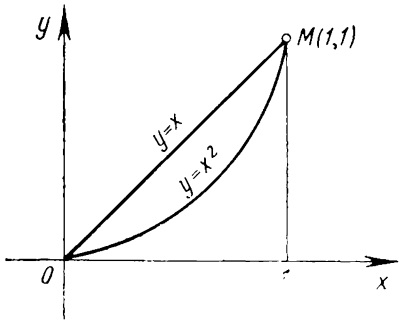

Fig. 1

In the following examples, find the distances between the given curves in the indicated intervals.

30. $f(x) = xe^{-x}$, $f_1(x) \equiv 0$, $[0, 2]$.

31. $f(x) = \sin 2x$, $f_1(x) = \sin x$, $\left[0, \frac{\pi}{2}\right]$.

32. $f(x) = x$, $f_1(x) = \ln x$, $[e^{-1}, e]$.

Let the curves $y = f(x)$ and $y = f_1(x)$ have continuous derivatives of order n on the interval $[a, b]$.

The *nth order distance between the curves* $y = f(x)$ and $y = f_1(x)$ is the largest of the maxima of the quantities

$$|f_1(x) - f(x)|,\ |f_1'(x) - f'(x)|,\ \ldots,\ |f_1^{(n)}(x) - f^{(n)}(x)|$$

on the interval $[a, b]$. We will denote this distance as follows:

$$\rho_n = \rho_n[f_1(x), f(x)] = \max_{0 \leqslant k \leqslant n} \max_{a \leqslant x \leqslant b} |f_1^{(k)}(x) - f^{(k)}(x)| \qquad (8)$$

The definition of the distance between curves given on page 30 is, in the sense of the new definition, a distance of the zeroth order.

Example 8. Find the first-order distance between the curves $f(x) = x^2$ and $f_1(x) = x^3$ on the interval $0 \leqslant x \leqslant 1$.

Solution. We find the derivatives of the given functions, $f'(x) = 2x$ and $f_1'(x) = 3x^2$, and we consider the functions

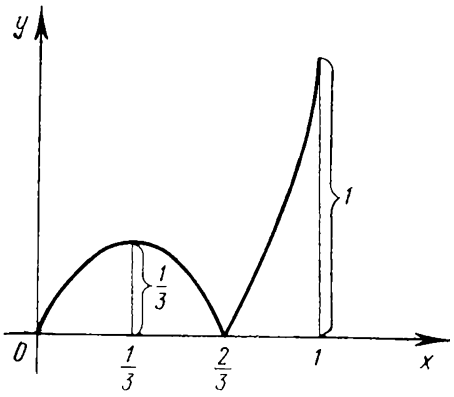

Fig. 2

$y_1(x) = x^2 - x^3$ and $y_2(x) = 2x - 3x^2$. Let us find their maximum values on the interval [0, 1]. We have $y_1' = 2x - 3x^2$. Equating this derivative to zero, we find the stationary points of the function $y_1(x)$: $x_1 = 0$, $x_2 = \frac{2}{3}$. Furthermore, $y_1|_{x=0} = 0$, $y_1|_{x=\frac{2}{3}} = \frac{4}{27}$. The value of $y_1(x)$ at the right endpoint is $y_1(1) = 0$, whence

$$\rho_0 = \max_{0 \leqslant x \leqslant 1} |x^3 - x^2| = \max_{0 \leqslant x \leqslant 1} (x^2 - x^3) = \frac{4}{27}$$

Let us now find the zeroth-order distance $\tilde{\rho}_0$ between the derivatives $f'(x) = 2x$ and $f_1'(x) = 3x^2$:

$$\tilde{\rho}_0 = \max_{0 \leqslant x \leqslant 1} |y_2'(x)| = \max_{0 \leqslant x \leqslant 1} |2x - 3x^2|$$

We construct the graph of the function $y = |2x - 3x^2|$ (Fig. 2). From the figure it is evident that $\tilde{\rho}_0 = 1$. Thus,

the first-order distance ρ_1 between the curves $f(x) = x^2$ and $f_1(x) = x^3$ is equal to

$$\rho_1 = \max(\rho_0, \tilde{\rho}_0) = 1$$

33. Find the first-order distance between the curves $f(x) = \ln x$ and $f_1(x) = x$ on the interval $[e^{-1}, e]$.

34. Find the second-order distance between the curves $f(x) = x$ and $f_1(x) = -\cos x$ on the interval $\left[0, \frac{\pi}{3}\right]$.

35. Find the 1001st-order distance between the curves $f(x) = e^x$ and $f_1(x) = x$ on the interval $[0, 1]$.

The *nth order ε-neighbourhood of a curve* $y = f(x)$ $(a \leqslant x \leqslant b)$ is defined as the collection of curves $y = f_1(x)$ whose nth order distances from the curve $y = f(x)$ are less than ε:

$$\rho_n = \rho_n [f(x), f_1(x)] < \varepsilon \tag{9}$$

A zeroth-order ε-neighbourhood is called a *strong ε-neighbourhood of the function* $y = f(x)$.

The strong ε-neighbourhood of the curve $y = f(x)$ consists of curves located in a strip of width 2ε about the curve $y = f(x)$.

A first-order ε-neighbourhood is called a *weak ε-neighbourhood of the function* $y = f(x)$.

2°. Continuity of a functional. A functional $J[y(x)]$ defined in a class M of functions $y(x)$ is said to be *continuous* for $y = y_0(x)$ in the sense of nth order proximity if for any number $\varepsilon > 0$ there exists a number $\eta > 0$ such that for all admissible functions $y = y(x)$ satisfying the conditions

$$|y(x) - y_0(x)| < \eta, \quad |y'(x) - y_0'(x)| < \eta, \quad \ldots, \quad |y^{(n)}(x) - y_0^{(n)}(x)| < \eta$$

the inequality $|J[y(x)] - J[y_0(x)]| < \varepsilon$ holds true. In other words, $|J[y(x)] - J[y_0(x)]| < \varepsilon$ if

$$\rho_n [y(x), y_0(x)] < \eta$$

A functional that is not continuous in the sense of nth order proximity will be called *discontinuous* in the sense of the indicated proximity. Putting

$$y^{(k)}(x) = y_0^{(k)}(x) + \alpha\omega^{(k)}(x) \qquad (k = 0, 1, 2, \ldots, n)$$

where α is some parameter and $\omega(x)$ is an arbitrary function in the class M, we note that

$$\lim_{\alpha \to 0} y^{(k)}(x) = y_0^{(k)}(x) \qquad (k = 0, 1, 2, \ldots, n)$$

and the definition of the continuity of the functional when $y(x) = y_0(x)$ may be written thus:

$$\lim_{\alpha \to 0} J[y_0(x) + \alpha\omega(x)] = J[y_0(x)]$$

Example 9. Show that the functional

$$J[y(x)] = \int_0^1 [y(x) + 2y'(x)]\, dx$$

defined in the space $C_1[0, 1]$ is continuous on the function $y_0(x) = x$ in the sense of first-order proximity.

Solution. Take an arbitrary number $\varepsilon > 0$ and show that there exists a number $\eta > 0$ such that $|J[y(x)] - J[x]| < \varepsilon$ as soon as $|y(x) - x| < \eta$ and $|y'(x) - 1| < \eta$. We have

$$|J[y(x)] - J[x]| = \left| \int_0^1 [y(x) + 2y'(x) - x - 2]\, dx \right|$$

$$\leqslant \int_0^1 |y(x) - x|\, dx + 2 \int_0^1 |y'(x) - 1|\, dx$$

We choose $\eta = \frac{\varepsilon}{3}$. Then for all $y(x) \in C_1[0, 1]$ for which

$$|y(x) - x| < \frac{\varepsilon}{3} \text{ and } |y'(x) - 1| < \frac{\varepsilon}{3}$$

we will have

$$|J[y(x)] - J[x]| < \varepsilon$$

Thus, for every $\varepsilon > 0$ there is an $\eta > 0$, for example, $\eta = \frac{\varepsilon}{3}$, such that as soon as $\rho_1[y(x), x] < \eta$, then $|J[y(x)] - J[x]| < \varepsilon$. By definition, this means that the given functional is continuous on the function $y_0 = x$ in the sense of first-order proximity. It is easy to see that this functional is continuous in the sense of first-order proximity on any curve $y(x) \in C_1[0, 1]$.

Example 10. Let us consider the functional

$$J[f(x)] = f'(x_0)$$

where the functions $f(x) \in C_1[a, b]$ and $x_0 \in [a, b]$.

This functional is discontinuous on any function $f(x)$ in the sense of zeroth-order proximity. Indeed, let $\varphi(x)$ be such that $\varphi'(x_0) = 1$ and $|\varphi(x)| < \eta$ on the interval $[a, b]$. We take the function $f(x) = f_0(x) + \varphi(x)$, where $f_0(x) \in C_1[a, b]$. Then $f'(x_0) = f_0'(x_0) + 1$. It is clear that $\rho[f(x), f_0(x)] < \eta$, that is, the curves $f(x)$ and $f_0(x)$ are close in the sense of zeroth-order proximity. At the same time, $J[f(x)] - J[f_0(x)] = 1$, that is, the values of the functional are not close for any zeroth-order proximity of the arguments $f(x)$ and $f_0(x)$.

To be more exact, there exists an $\varepsilon > 0$ (namely, $\varepsilon < 1$) such that no matter what $\eta > 0$, there will be functions $f(x)$ such that

$$\rho_0[f, f_0] < \eta \quad \text{and} \quad |J[f] - J[f_0]| \geqslant \varepsilon$$

This signifies the discontinuity of the functional $J[f]$ in the sense of zeroth-order proximity.

We will show that this functional is continuous in the sense of first-order proximity.

Let us take any $\varepsilon > 0$. We have

$$|J[f(x)] - J[f_0(x)]| = |f'(x_0) - f_0'(x_0)|$$

Clearly, if we take $\eta = \varepsilon$, then for $\rho_1[f(x), f_0(x)] < \eta$ we will have

$$|J[f(x)] - J[f_0(x)]| < \varepsilon$$

which is what we set out to prove. This example shows that, generally speaking, the continuity of a functional in the sense of nth order proximity does not imply continuity of the functional in the sense of proximity of a lower order.

Example 11. Let us consider the functional

$$J[y(x)] = \int_0^{\pi} y'^2(x)\, dx$$

defined in the space $C_1[0, \pi]$. We will show that this functional is discontinuous on the function $y_0(x) \equiv 0$ in the sense of zeroth-order proximity.

Indeed, let $y_0(x) \equiv 0$ on $[0, \pi]$ and $y_n(x) = \frac{\sin nx}{n}$. Then $\rho_0[y_0(x), y_n(x)] = \frac{1}{n}$ and $\rho_0 \to 0$ as $n \to \infty$.

On the other hand, the difference

$$J[y_n(x)] - J[y_0(x)] = \int_0^{\pi} \frac{\cos^2 nx}{n}\, dx = \frac{\pi}{2}$$

does not depend on n. Thus, as $n \to \infty$, $J[y_n(x)]$ does not tend to $J[y_0(x)] = 0$ and, hence, the given functional is discontinuous in the sense of zeroth-order proximity on the function $y_0(x) \equiv 0$.

We leave it to the reader to prove that the functional under consideration is continuous on the function $y_0(x) \equiv 0$ in the sense of first-order proximity.

Investigate the following functionals for continuity.

36. $J[y(x)] = y(x_0)$, where the functions $y(x) \in C[a, b]$ and $x_0 \in [a, b]$ in the sense of zeroth-order proximity.

37. $J[y(x)] = \max |y(x)|$, where the functions $y(x)$ are continuous on the interval $[a, b]$ in the sense of zeroth-order proximity.

38.

$$J[y(x)] = \begin{cases} 0 & \text{if } y(x) \text{ assumes at least one negative value,} \\ \frac{1}{2} & \text{if } y(x) \equiv 0, \\ 1 & \text{if } y(x) \geqslant 0 \text{ and } y(x) \not\equiv 0 \end{cases}$$

in the sense of zeroth-order proximity.

39. $J[y(x)] = \int_0^1 |y'(x)|\, dx$, where the functions $y(x)$ have continuous first derivatives on the interval $[0, 1]$: (a) in the sense of zeroth-order proximity; (b) in the sense of first-order proximity.

40. $J[y(x)] = \int_0^{\pi} \sqrt{1 + y'^2(x)}\, dx$ on the function $y_0(x) \equiv 0$, where the function $y(x) \in C_1[0, \pi]$: (a) in the sense

of zeroth-order proximity; (b) in the sense of first-order proximity.

41. $J[y(x)] = \int_0^{\pi} (1 + 2y'^2(x))\,dx$ on the function $y_0(x) \equiv 0$, where the function $y(x) \in C_1[0, \pi]$ in the sense of first-order proximity.

Example 12. Show that the functional

$$J[y(x)] = \int_0^1 x^3 \sqrt{1 + y^2(x)}\,dx$$

defined on the set of functions $y(x) \in C[0, 1]$ is continuous on the function $y_0(x) = x^2$ in the sense of zeroth-order proximity.

Solution. Put $y(x) = x^2 + \alpha\eta(x)$, where $\eta(x) \in C[0, 1]$ and α is arbitrarily small,

$$J[y(x)] = J[x^2 + \alpha\eta(x)] = \int_0^1 x^3 \sqrt{1 + (x^2 + \alpha\eta(x))^2}\,dx$$

$$= \int_0^1 x^3 \sqrt{1 + x^4 + 2\alpha x^2\eta(x) + \alpha^2\eta^2(x)}\,dx$$

Passing to the limit as $\alpha \to 0$, we get from this equation

$$\lim_{\alpha \to 0} J[y(x)] = \int_0^1 x^3 \sqrt{1 + x^4}\,dx = J[x^2]$$

and this signifies the continuity of the functional on the function $y_0 = x^2$.

Definition. Let M be a normed linear space of the functions $y(x)$.

The functional $L[y(x)]$ defined in the space M is called a *linear functional* if it satisfies the conditions:

(1) $$L[cy(x)] = c \cdot L[y(x)]$$

where c is an arbitrary constant,

(2) $$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$$

where $y_1(x) \in M$ and $y_2(x) \in M$.

For example, the functional

$$L[y(x)] = \int_a^b [y'(x) + y(x)]\,dx$$

defined in the space $C_1[a, b]$ is clearly a linear functional.

Here is another definition of the linearity of a functional.

A functional $L[y(x)]$ is said to be *linear* if it (1) is continuous, and (2) for any $y_1(x) \in M$ and $y_2(x) \in M$ satisfies the condition

$$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$$

42. Demonstrate that the two foregoing definitions of the linearity of a functional are equivalent.

43. Show that the functional $L[y(x)] = y(x_0)$ is linear.

44. Show that if $L[y(x)]$ is a linear functional and the ratio $\frac{L[y(x)]}{\|y(x)\|} \to 0$ as $\|y(x)\| \to 0$, then $L[y(x)] \equiv 0$.

3°. Variation of a functional. Let the functional $J[y(x)]$ be given on a set M of functions $y(x)$. The following quantity is termed the *increment of the functional* $J[y(x)]$, which corresponds to the increment $\delta y(x)$ in the argument:

$$\Delta J = \Delta J[y(x)] = J[y(x) + \delta y(x)] - J[y(x)] \quad (10)$$

$(\delta y(x) = \tilde{y}(x) - y(x)$, where $y(x) \in M$, $\tilde{y}(x) \in M)$

Example 13. Find the increment in the functional

$$J[y(x)] = \int_0^1 y(x)\,y'(x)\,dx$$

defined in the space $C_1[a, b]$ if $y(x) = x$, $y_1(x) = x^2$.

Solution. We have

$$\Delta J = J[x^2] - J[x] = \int_0^1 x^2 2x\,dx - \int_0^1 x \cdot 1 \cdot dx$$

$$= \int_0^1 (2x^3 - x)\,dx = 0$$

45. Find the increment in the functional considered in Example 13 by putting $y(x) = e^x$, $y_1(x) = 1$.

Definition. If the increment in the functional,

$$\Delta J = J[y(x) + \delta y] - J[y(x)]$$

can be represented as

$$\Delta J = L[y(x), \delta y] + \beta(y(x), \delta y) \| \delta y \|$$

where $L[y(x), \delta y]$ is a linear functional with respect to δy and $\beta(y(x), \delta y) \to 0$ as $\| \delta y \| \to 0$, then the portion of the increment of the functional that is linear with respect to δy, that is, $L[y(x), \delta y]$ is called the *variation of the functional* and is denoted by δJ. In this case the functional $J[y(x)]$ is said to be *differentiable* at the point $y(x)$.

46. Show that the variation δJ of the functional $J[y(x)]$ (if the variation exists) is defined in unique fashion.

Example 14. Show that the functional

$$J[y(x)] = \int_a^b y(x)\, dx$$

specified in the space $C[a, b]$ is differentiable at each point $y(x)$ of that space.

Solution.

$$\Delta J = J[y + \delta y] - J[y]$$

$$= \int_a^b [y(x) + \delta y(x)]\, dx - \int_a^b y(x)\, dx = \int_a^b \delta y(x)\, dx$$

Thus, $\Delta J = \int_a^b \delta y(x)\, dx$. This is a linear functional with respect to $\delta y(x)$. In the given case, the entire increment of the functional reduced to the linear functional with respect to $\delta y(x)$. This functional is differentiable at every point $y(x)$ and its variation $\delta J = \int_a^b \delta y(x)\, dx$.

47. Show that every continuous linear functional $J[y]$ is always differentiable.

Example 15. Show that the functional

$$J[y] = \int_a^b y^2(x)\,dx$$

defined in the space $C[a, b]$ is differentiable at every point $y(x)$.

Solution. We have

$$\Delta J = \int_a^b [y(x) + \delta y(x)]^2\,dx - \int_a^b y^2(x)\,dx$$

$$= \int_a^b 2y(x)\,\delta y(x)\,dx + \int_a^b (\delta y(x))^2\,dx \qquad (11)$$

The first integral in the right member of (11) is a linear functional with respect to $\delta y(x)$ for every fixed function $y(x)$. Let us estimate the second integral in the right-hand member of (11). We have

$$\int_a^b (\delta y(x))^2\,dx = \int_a^b |\delta y(x)|^2\,dx$$

$$\leqslant (\max_{a\leqslant x\leqslant b} |\delta y(x)|)^2 \int_a^b dx = (b-a)\,\|\delta y(x)\|^2$$

$$= ((b-a)\,\|\delta y\|)\cdot\|\delta y\|$$

The quantity $(b - a)\,\|\delta y\| \to 0$ as $\|\delta y\| \to 0$. Thus, the increment ΔJ of the functional can be represented in the form of a sum of $L[y, \delta y]$ and an extra portion having second order of smallness with respect to $\|\delta y\|$. According to the definition, the given functional is differentiable at a point of $y(x)$ and its variation is

$$\delta J = 2\int_a^b y(x)\,\delta y(x)\,dx$$

48. For the functional $J[y(x)] = \int_0^1 y^2(x)\,dx$ put $y = 2x$, $\delta y = \alpha x^2$ and compare δJ with ΔJ for $\alpha = 1, -0.1, 0.01$.

49. For the functional $J[y(x)] = \int_0^1 xy^3(x)\,dx$ put $y = e^x$, $\delta y = \alpha x$. Compare ΔJ with δJ for $\alpha = 1, 0.1, 0.01$.

50. Verify the differentiability of the following functionals:

(1) $J[y] = y(a)$ in the space $C[a, b]$.
(2) $J[y] = y(a)$ in the space $C_1[a, b]$.
(3) $J[y] = \sqrt{1 + y'^2(a)}$ in the space $C_1[a, b]$.
(4) $J[y] = |y(a)|$ in the space $C[a, b]$.

51. Show that the functional $J^2[y]$ is differentiable if $J[y]$ is differentiable. Write down the variation of $J^2[y]$.

52. Show that the functional

$$J[y] = \int_a^b f(x, y(x))\,dx$$

defined in the space $C[a, b]$, where $f(x, y)$ is a continuous function of its arguments having continuous partial derivatives up to order two inclusive in the domain $a \leqslant x \leqslant b$, $-\infty < y < +\infty$, is differentiable and its variation is of the form

$$\delta J = \int_a^b \frac{\partial f(x, y)}{\partial y}\,\delta y(x)\,dx$$

Example 16. Consider the functional

$$J[y] = \int_a^b f(x, y(x), y'(x))\,dx$$

defined in the space $C_1[a, b]$ of continuous functions $y(x)$ on the interval $[a, b]$, the functions having continuous first derivatives. The function $f(x, y, y')$ is continuous with respect to all its arguments and has continuous partial derivatives up to order two inclusive in the domain

$$a \leqslant x \leqslant b, \quad -\infty < y < +\infty, \quad -\infty < y' < +\infty$$

Let us find the increment ΔJ of the functional that corresponds to the increment $\delta y(x)$ of the argument, where

$\delta y(x) \in C_1[a, b]$. We have

$$\Delta J[y(x)] = \int_a^b [f(x, y+\delta y, y'+\delta y') - f(x, y, y')]\, dx \qquad (12)$$

By Taylor's formula,

$$f(x, y+\delta y, y'+\delta y') - f(x, y, y') = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + R(x, y, y', \delta y, \delta y') \qquad (13)$$

where $R(x, y, y', \delta y, \delta y')$ is the remainder term of Taylor's formula. Putting (13) into (12), we get

$$\Delta J[y(x)] = \int_a^b \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right) dx + \int_a^b R(x, y, y', \delta y, \delta y')\, dx \qquad (14)$$

The first term on the right of (14) is linear with respect to δy and $\delta y'$. Suppose all second partial derivatives of the function $f(x, y, y')$ with respect to y and y' do not exceed, in absolute value, a constant $M > 0$ in the domain bounded with respect to y and y'. Then the following estimate holds true:

$$\int_a^b |R(x, y, y', \delta y, \delta y')|\, dx \leqslant 2M \int_a^b \|\delta y\|^2\, dx = 2M(b-a)\|\delta y\|^2$$

Here, $\|\delta y\| = \max_{a \leqslant x \leqslant b} (|\delta y|, |\delta y'|)$. Thus, the second term in the right member of (14) is of second order relative to $\|\delta y\|$. Consequently, by definition, the functional $J[y]$ is differentiable in the space $C_1[a, b]$ and its variation is of the form

$$\delta J[y] = \int_a^b \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right) dx \qquad (15)$$

Example 17. Find the variation of the functional

$$J[y] = \int_{-1}^{1} (y'e^y + xy^2)\,dx$$

Solution. The function $f(x, y, y') = y'e^y + xy^2$ is obviously continuous in the variables x, y and y' jointly, and has partial derivatives of all orders with respect to y and y' that are bounded in any restricted range of the variables y and y'. Therefore the given functional is differentiable in the space $C_1[-1, 1]$ and its variation is, by formula (15), equal to

$$\delta J = \int_{-1}^{1} [(y'e^y + 2xy)\,\delta y + e^y \delta y']\,dx$$

53. For the functional

$$J[y(x)] = \int_{1}^{e} (y'y + xy'^2)\,dx$$

put $y = \ln x$, $\delta y = \frac{k(x-1)}{e-1}$ and compare $\Delta J[y(x)]$ with $\delta J[y(x)]$ for $k = 1,\ 0.1,\ 0.01$.

54. For the functional

$$J[y(x)] = \int_{0}^{1} (x^2y'^2 - y^2)\,dx$$

put $y = x^2$, $\delta y = kx^3$. Compare $\Delta J[y(x)]$ with $\delta J[y(x)]$ for $k = 1,\ 0.1,\ 0.01$.

55. For the functional $J[y(x)] = \int_{0}^{\pi} y'^2 \sin x\,dx$ put $y = \sin x$, $\delta y = k \cos x$. Compare $\Delta J[y(x)]$ with $\delta J[y(x)]$ for $k = -1,\ 0.3,\ 0.03$.

56. Show that if the function $f(x, z_1, z_2, \ldots, z_{m+1})$ has continuous second derivatives with respect to all arguments in the domain $a \leqslant x \leqslant b$, $-\infty < z_k < +\infty$ $(k =$

$= 1, 2, \ldots, m + 1)$, then the functional

$$J[y] = \int_a^b f[x, y(x), y'(x), \ldots, y^{(m)}(x)]\, dx$$

is differentiable in the space $C_m[a, b]$ and its variation is of the form

$$\delta J = \int_a^b \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' + \ldots + \frac{\partial f}{\partial y^{(m)}} \delta y^{(m)} \right) dx \qquad (16)$$

4°. **Second definition of the variation of a functional.** The *variation of the functional* $J[y(x)]$ at the point $y = y(x)$ is defined as the value of the derivative of the functional $J[y(x) + \alpha \delta y(x)]$ with respect to the parameter α when $\alpha = 0$:

$$\delta J = \frac{\partial}{\partial \alpha} J[y(x) + \alpha \delta y(x)]\,|_{\alpha=0} \qquad (17)$$

If the variation of a functional exists as the principal linear part of its increment, that is, in the sense of the first definition, then the variation also exists as the value of the derivative with respect to the parameter α when $\alpha = 0$, and these variations coincide.

Example 18. Using the second definition, find the variation of the functional

$$J[y(x)] = \int_a^b y^2(x)\, dx$$

Solution. The variation of this functional in the sense of the first definition is equal to

$$\delta y = 2 \int_a^b y(x)\, \delta y(x)\, dx$$

(see Example 15). Let us find the variation of the functional $J[y]$ using the second definition of a variation. We have

$$J[y(x) + \alpha \delta y(x)] = \int_a^b [y(x) + \alpha \delta y(x)]^2\, dx$$

Then

$$\frac{\partial}{\partial \alpha} J[y+\alpha\delta y] = 2\int_a^b (y+\alpha\delta y)\,\delta y\,dx$$

and, consequently,

$$\delta J = \frac{\partial}{\partial \alpha} J[y+\alpha\delta y]\,|_{\alpha=0} = 2\int_a^b y\delta y\,dx$$

The variations of the functional in the sense of the first and second definitions coincide.

Find the variations of the following functionals in the indicated spaces in the sense of the second definition.

57. $J[y] = \int_a^b (x+y)\,dx.$

58. $J[y] = \int_a^b (y^2 - y'^2)\,dx.$

59. $J[y] = y^2(0) + \int_0^1 (xy + y'^2)\,dx.$

60. $J[y] = \int_0^\pi y' \sin y\,dx.$

61. Find the variation of the functional

$$J[y_1, y_2, \ldots, y_n] = \int_a^b f(x, y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x))\,dx$$

where f is a continuous function of its arguments and has continuous partial derivatives with respect to all its arguments in some bounded range G of these arguments.

Note. The second definition of a variation of a functional is somewhat broader than the first in the sense that there are functionals from the increments of which it is not possible to solate a principal linear part, but the variation

exists in the sense of the second definition. We will demonstrate this fact using the example of functions for which the formulated assertion is equivalent to the fact that the existence of derivatives in any direction is not sufficient for the existence of a differential of the function.

Let

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} = \frac{\rho}{2} \sin 2\varphi \qquad (x^2 + y^2 \neq 0)$$

where ρ and φ are the polar coordinates of the point (x, y). The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point and at the origin are equal to zero, but the differential df does not exist at the origin. Indeed, given the existence of df, the gradient of the function f at the origin would in this case be equal to zero, and therefore a derivative in any direction, $\frac{df(0, 0)}{dl}$, would be equal to zero. Yet, as can readily be seen,

$$\frac{df(0, 0)}{dl} = \frac{1}{2} \sin 2\varphi$$

which is, generally speaking, different from zero. Here, φ is the angle formed by the vector l with the x-axis and the derivative in the direction of l coincides with the derivative with respect to ρ.

5°. **The second variation of a functional.** A functional $J[x, y]$ dependent on two elements x and y (lying in a certain linear space) is called a *bilinear functional* if, for a fixed x, it is a linear functional of y and, for a fixed y, it is a linear functional of x. Thus, the functional $J[x, y]$ is bilinear if

$$J[\alpha_1 x_1 + \alpha_2 x_2, y] = \alpha_1 J[x_1, y] + \alpha_2 J[x_2, y],$$
$$J[x, \beta_1 y_1 + \beta_2 y_2] = \beta_1 J[x, y_1] + \beta_2 J[x, y_2]$$

Putting $y = x$ in a bilinear functional, we get the expression $J[x, x]$, which is called a *quadratic functional.*

A bilinear functional in a finite-dimensional space is termed a *bilinear form.*

A quadratic functional $J[x, x]$ is said to be *positive definite* if $J[x, x] > 0$ for any nonzero element x.

Examples.

(1) The expression

$$J[x, y] = \int_a^b A(t)\, x(t)\, y(t)\, dt$$

where $A(t)$ is a fixed continuous function, is a bilinear functional, and $\int_a^b A(t)\, x^2(t)\, dt$ is a quadratic functional in the space $C[a, b]$; also, if $A(t) > 0$ for all $t \in [a, b]$, then this quadratic functional will be positive definite.

(2) The expression

$$\int_a^b [A(t)\, x^2(t) + B(t)\, x(t)\, x'(t) + C(t)\, x'^2(t)]\, dt$$

is an example of a quadratic functional defined for all functions in the space $C_1[a, b]$.

(3) The integral

$$\int_a^b \int_a^b K(s, t)\, x(s)\, y(t)\, ds\, dt$$

where $K(s, t)$ is a fixed function of two variables, is a bilinear functional in $C[a, b]$.

Definition. Let $J[y]$ be a functional defined in some normed linear space.

We will say that the functional $J[y]$ has a second variation if its increment $\Delta J = J[y + \delta y] - J[y]$ may be written in the form

$$\Delta J = L_1[\delta y] + \frac{1}{2} L_2[\delta y] + \beta \| \delta y \|^2 \qquad (18)$$

where $L_1[\delta y]$ is a linear functional, $L_2[\delta y]$ is a quadratic functional, and $\beta \to 0$ as $\| \delta y \| \to 0$.

We will call the quadratic functional $L_2[\delta y]$ the *second variation* (second differential) of the functional $J[y]$ and denote it by $\delta^2 J$.

The second variation of a functional (if it exists) is defined uniquely.

Example 19. Find the second variation of the functional

$$J[y] = \int_0^1 (xy^2 + y'^3)\, dx$$

defined in the space $C_1[0, 1]$ of functions $y(x)$.

Solution. We have

$$\Delta J = J[y + \delta y] - J[y]$$

$$= \int_0^1 [x(y+\delta y)^2 + (y' + \delta y')^3 - xy^2 - y'^3]\, dx$$

$$= \int_0^1 [2xy\,\delta y + x(\delta y)^2 + 3y'^2\,\delta y' + 3y'(\delta y')^2$$

$$+ (\delta y')^3]\, dx = \int_0^1 (2xy\,\delta y + 3y'^2\,\delta y')\, dx$$

$$+ \int_0^1 [x(\delta y)^2 + 3y'(\delta y')^2]\, dx + \int_0^1 (\delta y')^3\, dx \qquad (19)$$

For a fixed $y(x)$, the first term in the right-hand member of (19) is a functional linear with respect to $\delta y(x)$; the second term of the right-hand member is a quadratic functional. Finally, the third term of the right-hand member allows for the obvious estimate

$$\left| \int_0^1 (\delta y')^3\, dx \right| \leqslant (\max |\delta y'|)^2 \int_0^1 |\delta y'|\, dx \leqslant \int_0^1 |\delta y'|\, dx \, \| \delta y \|^2$$

(the norm in the sense of the space $C_1[0, 1]$), whence it is seen that this term can be represented in the form $\beta \cdot \| \delta y \|^2$, where $\beta \to 0$ as $\| \delta y \| \to 0$. According to the definition, the given functional has a second variation $\delta^2 J$ and it is equal to

$$\delta^2 J = 2 \int_0^1 [x(\delta y)^2 + 3y'(\delta y')^2]\, dx$$

62. Prove that a quadratic functional is differentiable and find its second variation.

63. Write down the second variation of the functional $e^{F(y)}$, where $F(y)$ is a twice differentiable functional.

64. Show that functionals of the form

$$J[y] = \int_a^b F(x, y, y')\, dx$$

in the space $C_1[a, b]$ are twice differentiable if the integrand F has continuous derivatives up to order three inclusive. Find the expression for the second variation.

We introduce the function $\Phi(\alpha) = J[y + \alpha\delta y]$. The second variation $\delta^2 J$ of the functional $J[y]$ is also defined in terms of the second derivative of the function $\Phi(\alpha)$ at the point $\alpha = 0$:

$$\delta^2 J = \frac{d^2\Phi(\alpha)}{d\alpha^2}\Big|_{\alpha=0}$$

Both of these definitions coincide for functionals of the integral type, which are the ones that we will be considering for the most part.

Find the second variations.

65. $J[y] = \int_a^b F(x, y, y', \ldots, y^{(m)})\, dx.$

66. $J[y] = \iint_G F(x, y, z, z_x, z_y)\, dx\, dy.$

67. $J[y_1, \ldots, y_n] = \int_a^b F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx.$

6°. Extremum of a functional. Necessary condition for an extremum. We say that a functional $J[y(x)]$ attains a maximum on a curve $y = y_0(x)$ if the values of the functional $J[y(x)]$ on any curve close to $y = y_0(x)$ do not exceed $J[y_0(x)]$, that is,

$$\Delta J = J[y(x)] - J[y_0(x)] \leqslant 0$$

If $\Delta J \leqslant 0$, and $\Delta J = 0$ only when $y(x) = y_0(x)$, then we say that a strict maximum is attained on the curve $y = y_0(x)$.

The curve $y = y_0(x)$ on which a minimum is attained is defined in similar fashion. In this case $\Delta J \geqslant 0$ on all curves close to the curve $y = y_0(x)$.

Example 20. Show that the functional

$$J[y(x)] = \int_0^1 (x^2 + y^2)\, dx$$

attains a strict minimum on the curve $y(x) \equiv 0$.

Solution. For any function $y(x)$ continuous on $[0, 1]$ we have

$$\Delta J = J[y(x)] - J[0] = \int_0^1 (x^2 + y^2)\, dx - \int_0^1 x^2\, dx = \int_0^1 y^2\, dx \geqslant 0$$

equality occurring only when $y(x) \equiv 0$.

Strong and weak extrema. We say that a functional $J[y(x)]$ attains a *strong relative maximum* on a curve $y = y_0(x)$ if for all admissible curves $y = y(x)$ located in an ε-neighbourhood of zeroth order of the curve $y = y_0(x)$ we have

$$J[y(x)] \leqslant J[y_0(x)]$$

A *strong relative minimum* of the functional is defined similarly.

We say that the functional $J[y(x)]$ attains a *weak relative maximum* on a curve $y = y_0(x)$ if for all admissible curves $y = y(x)$ located in an ε-neighbourhood of the first order of the curve $y = y_0(x)$ we have

$$J[y(x)] \leqslant J[y_0(x)]$$

A *weak relative minimum* of the functional is defined similarly.

The maxima and minima (strong and weak) of a functional $J[y]$ are termed *relative extrema*.

Every strong extremum is at the same time a weak extremum but not conversely.

An extremum of a functional $J[y]$ on the entire collection of functions on which it is defined is called an *absolute extremum.*

Every absolute extremum is a weak and strong relative extremum, but not every relative extremum is an absolute extremum.

Example 21. Consider the functional

$$J[y(x)] = \int_0^{\pi} y^2 (1 - y'^2)\, dx$$

in the space of functions $y(x) \in C_1[0, \pi]$ that satisfy the condition $y(0) = y(\pi) = 0$. The interval $[0, \pi]$ on the x-axis yields a weak minimum of J. Indeed, for $y \equiv 0$ we have $J = 0$, whereas for the curves located in a first-order ε-neighbourhood of this interval, where ε is any positive number less than unity, we have $|y'| < 1$, so that the integrand is positive for $y \neq 0$ and, hence, the functional vanishes only when $y = 0$. Hence, a weak minimum is attained on the function $y = 0$.

A strong minimum is not attained. It suffices to put

$$y(x) = \frac{1}{\sqrt{n}} \sin nx$$

Then

$$J[y(x)] = \frac{1}{n} \int_0^{\pi} \sin^2 nx\, (1 - n \cos^2 nx)\, dx$$

$$= \frac{1}{n} \int_0^{\pi} \sin^2 nx\, dx - \frac{1}{4} \int_0^{\pi} \sin^2 2nx\, dx = \frac{\pi}{2n} - \frac{\pi}{8}$$

and for n sufficiently large for our curves $J < 0$. On the other hand, for n sufficiently large, all these curves lie within an arbitrarily small neighbourhood of zeroth order of the curve $y = 0$. Thus, a strong minimum is not attained for $y = 0$.

Example 22 (Weierstrass). Consider the functional

$$J[y] = \int_{-1}^{1} x^2 y'^2\, dx, \quad y(-1) = -1, \quad y(1) = 1$$

We have $J[y] \geqslant 0$ on the interval $[-1, 1]$, and $J[y] = 0$ only when $y'(x) \equiv 0$, that is $y(x) \equiv C$ = constant. The function $y(x) \equiv C$ lies in the class $C_1[-1, 1]$ of functions having on the interval $[-1, 1]$ a continuous first derivative but does not satisfy the given boundary conditions. Hence, $J[y] > 0$ for all $y(x) \in C_1[-1, 1]$ that satisfy the conditions $y(-1) = -1$, $y(1) = 1$. Thus, the functional has a lower bound but it is not attained on the curves $y(x) \in C_1[-1, 1]$. Indeed, let us consider the one-parameter family of curves

$$y_\alpha(x) = \frac{\arctan \frac{x}{\alpha}}{\arctan \frac{1}{\alpha}}, \qquad \alpha > 0$$

These curves satisfy the boundary conditions $y_\alpha(-1) = -1$, $y_\alpha(1) = 1$. In the limit, as $\alpha \to 0$, we get the function

$$\tilde{y}(x) = \begin{cases} -1 \text{ if } -1 \leqslant x < 0, \\ 0 \text{ if } x = 0, \\ +1 \text{ if } 0 < x \leqslant 1 \end{cases}$$

or $\tilde{y}(x) = \operatorname{sgn} x$ (Fig. 3).

This function belongs to the class of functions that are piecewise differentiable on the interval $[-1, 1]$.

We have

$$J[y_\alpha] = \int_{-1}^{1} \frac{\alpha x^2\, dx}{(\alpha^2 + x^2) \arctan^2 \frac{1}{\alpha}} = \frac{2\alpha}{\arctan^2 \frac{1}{\alpha}} \int_0^1 \frac{x^2\, dx}{\alpha^2 + x^2}$$

$$= \frac{2\alpha}{\arctan^2 \frac{1}{\alpha}} \left(1 - \alpha \arctan \frac{1}{\alpha}\right)$$

It is clear that $J[y_\alpha] \to 0$ as $\alpha \to 0$. On the limiting function $\tilde{y}(x)$ that satisfies the boundary conditions $\tilde{y}(-1) = -1$, $\tilde{y}(1) = 1$, the functional $J[y]$ assumes a value equal to zero: $J[\tilde{y}] = 0$.

Thus, the functional $J[y]$ attains its minimum on the curve $\tilde{y}(x) = \operatorname{sgn} x$, which belongs to the class of functions

that are piecewise differentiable on the interval $[-1, 1]$ but do not lie in the class $C_1[-1, 1]$.

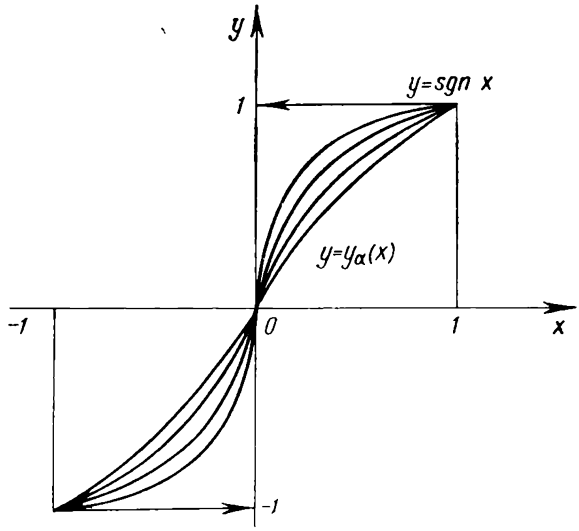

Fig. 3

Theorem (necessary condition for an extremum of a functional). *If a differentiable functional* $J[y(x)]$ *attains an extremum when* $y = y_0(x)$, *where* $y_0(x)$ *is an interior point of the domain of definition of the functional, then for* $y = y_0(x)$ *we have*

$$\delta J[y_0(x)] = 0 \tag{20}$$

Functions for which $\delta J = 0$ will be called *stationary functions.*

Using the necessary condition of an extremum (20) and the basic lemmas of the calculus of variations (for instance [16]), find the functional equations for determining the stationary functions of the following functionals.

68. $J[\varphi] = \int_a^b \int_a^b K(s, t)\,\varphi(s)\,\varphi(t)\,ds\,dt + \int_a^b \varphi^2(s)\,ds - 2\int_a^b \varphi(s)\,f(s)\,ds$

where $K(s, t)$ is a given continuous symmetric function of s and t in the domain $D\begin{Bmatrix} a \leqslant s \leqslant b \\ a \leqslant t \leqslant b \end{Bmatrix}$; $f(s)$ is a given continuous function on $[a, b]$; $\varphi(s)$ is the desired continuous functional argument.

69. $$J[\varphi] = \int_{-\infty}^{+\infty} [p(x)\varphi'^2(x) + 2\varphi(x+1)\varphi(x-1) - \varphi^2(x) - 2\varphi(x) f(x)]\, dx$$

where the functional argument $\varphi(x)$ is continuous and has piecewise continuous derivatives throughout the interval $-\infty < x < +\infty$; $p(x)$ has a continuous derivative and $f(x)$ is continuous.

70. $$J[\varphi] = \int_{x_0}^{x_1} [p(x)\varphi'^2 + q(x)\varphi^2(x) - 2\varphi(x) f(x)]\, dx,$$
$$\varphi(x_0) = \varphi_0, \quad \varphi(x_1) = \varphi_1$$

where $p(x)$ has a continuous derivative, $q(x)$ and $f(x)$ are continuous, and the functional argument $\varphi(x)$ is twice continuous differentiable.

4. AN ELEMENTARY PROBLEM IN THE CALCULUS OF VARIATIONS. EULER'S EQUATION

Let the function $F(x, y, y')$ have continuous partial derivatives up to order two inclusive with respect to all arguments.

From among all the functions $y(x)$ having a continuous derivative and satisfying the boundary conditions

$$y(a) = A, \quad y(b) = B \tag{1}$$

find the function which gives the functional

$$J[y(x)] = \int_a^b F(x, y, y')\, dx \tag{2}$$

a weak extremum. In other words, the simplest problem of the calculus of variations consists in finding the weak extremum of a functional of the form (2) on the set of all smooth curves connecting two given points $P_1(a, A)$ and $P_2(b, B)$.

Theorem 1. *For the functional* (2), *defined on the set of functions* $y = y(x)$ *having a continuous first derivative and satisfying the boundary conditions* (1), *to attain an extremum on the given function* $y(x)$, *it is necessary** *that this function satisfy the Euler equation*

$$F_y - \frac{d}{dx} F_{y'} = 0 \tag{3}$$

The integral curves of Euler's equation are called *extremals* (Lagrangian curves). Euler's equation in expanded form looks like this:

$$y''(x) F_{y'y'} + y'(x) F_{yy'} + F_{xy'} - F_y = 0 \qquad (F_{y'y'} \neq 0) \tag{4}$$

Equation (4) is a second-order differential equation so that its general solution must depend on two arbitrary constants. Generally speaking, the values of these constants are determined from the boundary conditions (1).

The extremum of functional (2) can be realized only on those extremals which satisfy the conditions (1).

The boundary-value problem

$$\left.\begin{aligned} F_y - \frac{d}{dx} F_{y'} = 0, \\ y(a) = A,\ y(b) = B \end{aligned}\right\} \tag{5}$$

does not always have a solution, and if a solution exists, it may not be unique.

Example 1. On what curves can the functional

$$J[y(x)] = \int_1^2 (y'^2 - 2xy)\, dx;\ y(1) = 0,\ y(2) = -1$$

attain an extremum?

* This condition is necessary for a weak extremum. Since any strong extremum is at the same time a weak extremum, it follows that any condition necessary for a weak extremum is also necessary for a strong extremum.

Solution. Here, $F(x, y, y') = y'^2 - 2xy$ so that Euler's equation is of the form $y'' + x = 0$. The general solution of Euler's equation is

$$y(x) = -\frac{x^3}{6} + C_1 x + C_2$$

The boundary conditions yield a system of linear equations for determining C_1 and C_2:

$$\left.\begin{aligned} C_1 + C_2 &= \frac{1}{6}, \\ 2C_1 + C_2 &= \frac{2}{6} \end{aligned}\right\}$$

whence $C_1 = \frac{1}{6}$, $C_2 = 0$. Hence, an extremum can only be attained on the curve

$$y = \frac{x}{6}(1 - x^2)$$

Example 2. Find the extremals of the functional

$$J[y(x)] = \int_1^3 (3x - y)\, y\, dx$$

that satisfy the boundary conditions $y(1) = 1$, $y(3) = 4\frac{1}{2}$.

Solution. Euler's equation has the form $3x - 2y = 0$, whence $y(x) = \frac{3}{2}x$.

Since the extremal $y = \frac{3}{2}x$ does not satisfy the condition $y(1) = 1$, the given variational problem does not have a solution.

Example 3. Find the extremals of the functional

$$J[y(x)] = \int_0^{2\pi} (y'^2 - y^2)\, dx$$

that satisfy the boundary conditions $y(0) = 1$, $y(2\pi) = 1$.

Solution. Euler's equation is of the form $y'' + y = 0$. The general solution is

$$y(x) = C_1 \cos x + C_2 \sin x$$

Using the boundary conditions, we get

$$y(x) = \cos x + C \sin x$$

where C is an arbitrary constant.

Thus, the variational problem has an infinitude of solutions.

Find the extremals of the following functionals.

71. $J[y] = \int_{-1}^{0} (12xy - y'^2)\,dx;\ y(-1) = 1,\ y(0) = 0.$

72. $J[y] = \int_{1}^{2} (y'^2 + 2yy' + y^2)\,dx;\ y(1) = 1,\ y(2) = 0.$

73. $J[y] = \int_{0}^{1} \sqrt{y(1 + y'^2)}\,dx;\ y(0) = y(1) = \frac{1}{\sqrt{2}}.$

74. $J[y] = \int_{0}^{1} yy'^2\,dx;\ y(0) = 1,\ y(1) = \sqrt[3]{4}.$

75. $J[y] = \int_{0}^{\pi} (4y \cos x + y'^2 - y^2)\,dx;\ y(0) = 0,\ y(\pi) = 0.$

76. $J[y] = \int_{0}^{1} (y'^2 - y^2 - y)\,e^{2x}\,dx;\ y(0) = 0,\ y(1) = e^{-1}.$

77. $J[y] = \int_{-1}^{1} (y'^2 - 2xy)\,dx;\ y(-1) = -1,\ y(1) = 1.$

78. $J[y] = \int_{-1}^{0} (y'^2 - 2xy)\,dx;\ y(-1) = 0,\ y(0) = 2.$

79. $J[y] = \int_{1}^{e} (xy'^2 + yy')\,dx;\ y(1) = 0,\ y(e) = 1.$

The Euler equation (3) for the functional (2) is a second-order differential equation so that the solution $y = y(x)$ of Euler's equation must have a second derivative $y''(x)$.

However, there are cases where the function on which the functional $J[y] = \int_a^b F(x, y, y')\,dx$ attains an extremum is not twice differentiable.

Example 4. The functional

$$J[y(x)] = \int_{-1}^{1} y^2(x)\,(1 - y'(x))^2\,dx$$

with boundary conditions

$$y(-1) = 0,\ y(1) = 1$$

attains its minimum, equal to zero, on the function

$$v(x)\begin{cases} 0 \text{ when } x \leqslant 0, \\ x \text{ when } x > 0 \end{cases}$$

Although the function $v(x)$ does not have a second derivative, it satisfies the appropriate Euler equation.

Indeed, since $F(x, y, y') = y^2(1 - y')^2$, by putting $y = v(x)$ we get Euler's equation

$$2v(1 - v')^2 + \frac{d}{dx}[2v^2(1 - v')] = 0 \qquad (6)$$

But according to the definition of the function $v(x)$ we will have, on $[-1, 1]$, $F_{v'} = -2v^2(1 - v') \equiv 0$ and, hence, also $\frac{d}{dx} F_{v'} = 0$; and although the Euler equation (6) is formally of the second order and $v''(x)$ does not exist, the substitution of $v(x)$ into the Euler equation converts it to an identity.

Theorem 2. *Let $y = y(x)$ be a solution of the Euler equation*

$$F_y - \frac{d}{dx} F_{y'} = 0$$

If the function $F(x, y, y')$ has continuous partial derivatives up to order two inclusive, then at all points (x, y) at which

$$F_{y'y'}(x, y(x), y'(x)) \neq 0 \qquad (7)$$

the function $y = y(x)$ has a continuous second derivative.

Corollary. *The extremal* $y = y(x)$ *can have a salient point only at those points where* $F_{y'y'} = 0$.

Thus, in Example 4, $F_{y'y'} = 2y^2$ vanishes at points on the x-axis; the extremal has a salient point at the point $x = 0$.

Theorem 3 (Bernstein). *Suppose we have an equation*

$$y'' = F(x, y, y') \tag{8}$$

If the functions F, F_y, $F_{y'}$ *are continuous at each endpoint* (x, y) *for every finite* y' *and if there exist a constant* $k > 0$ *and functions*

$$\alpha = \alpha(x, y) \geqslant 0, \quad \beta = \beta(x, y) \geqslant 0$$

bounded in every finite portion of the plane such that

$$F_y(x, y, y') > k, \tag{9}$$

$$|F(x, y, y')| \leqslant \alpha y'^2 + \beta \tag{10}$$

then one and only one integral curve $y = \varphi(x)$ *of equation* (8) *passes through any two points* (a, A) *and* (b, B) *of the plane that have distinct abscissas* $(a \neq b)$.

Example 5. Prove that through any two points of the plane with distinct abscissas there passes one and only one extremal of the functional

$$\int e^{-2y^2}(y'^2 - 1)\,dx$$

Solution. The Euler equation of the given functional is of the form

$$y'' = 2y(1 + y'^2)$$

and Theorem 3 is applicable. Indeed, in the given case,

$$F(x, y, y') = 2y(1 + y'^2) \text{ and}$$
$$F_y = 2(1 + y'^2) \geqslant 2 = k$$

Furthermore,

$$|F(x, y, y')| = |2y(1 + y'^2)|$$
$$\leqslant 2|y|y'^2 + 2|y|$$

so that $\alpha = \beta = 2|y| \geqslant 0$.

Example 6. Show that it is not possible to pass an extremal of the functional

$$J[y] = \int (y^2 + \sqrt{1+y'^2})\, dx$$

through just any two points of a plane having distinct abscissas.

Solution. The Euler equation is of the form

$$y'' = 2y\,(1 + y'^2)^{3/2} \tag{11}$$

and Theorem 3 is not applicable since the condition (10) is not fulfilled [$F(x, y, y')$ increases faster with respect to y' than the second power of y']. However, this does not yet imply that it is not possible to draw an extremal through just any two points with distinct abscissas.

Putting $y' = p$, $y'' = p\frac{dp}{dy}$ in equation (11), we get

$$p\frac{dp}{dy} = 2y\,(1 + p^2)^{3/2}$$

or

$$\frac{p\,dp}{(1+p^2)^{3/2}} = 2y\,dy$$

Integrating, we find $-\frac{1}{\sqrt{1+p^2}} = y^2 - C$ or

$$(C - y^2)\sqrt{1+y'^2} = 1$$

whence

$$\frac{dy}{dx} = \frac{\sqrt{1-(y^2-C)^2}}{y^2-C} \tag{12}$$

where C is a real constant.

It is easy to verify that for all values of y that satisfy, for example, the conditions $0 \leqslant y \leqslant b$, where $b > \sqrt{2}$, there is no admissible value of the constant C for which the right member of (12) will be real.

80. Show that one and only one extremal of the functional

$$J[y(x)] = \int \sqrt{1+y^2+y'^2}\, dx$$

passes through any two points in the plane.

Example 7. Prove that any equation

$$y''(x) = f(x, y, y') \tag{13}$$

is an Euler equation for some functional

$$J[y(x)] = \int F(x, y, y')\,dx \tag{14}$$

(1) How is the function $F(x, y, y')$ determined from the function $f(x, y, y')$?

(2) Find all functionals for which the straight lines

$$y(x) = C_1x + C_2$$

are extremals.

Solution. We will seek the functional for which the Euler equation

$$F_y - F_{y'x} - F_{y'y}y' - F_{y'y'}y'' = 0 \tag{15}$$

coincides with equation (13). This means that there must be an identity with respect to x, y, y':

$$F_y - F_{y'x} - F_{y'y}\cdot y' - F_{y'y'}\cdot f(x, y, y') \equiv 0$$

Differentiating this identity with respect to y', we get

$$F_{y'y'x} + F_{y'y'y}\cdot y' + F_{y'y'y'}\cdot f + F_{y'y'}\cdot f_{y'} = 0$$

Put $u = F_{y'y'}$. Then for the function u we get a partial differential equation:

$$\frac{\partial u}{\partial x} + y'\frac{\partial u}{\partial y} + f\frac{\partial u}{\partial y'} + f_{y'}\cdot u = 0 \tag{16}$$

Thus, finding the functional, that is, finding the function $F(x, y, y')$, reduces to integrating the linear partial differential equation (16) and to subsequent quadrature.

Let us consider the second question. In this case, the Euler equation must be of the form $y'' = 0$ and for the function u we get—by virtue of (16)—the equation

$$\frac{\partial u}{\partial x} + y'\frac{\partial u}{\partial y} = 0 \tag{17}$$

Let us integrate this equation.

The characteristic equation is of the form

$$\frac{dx}{1} = \frac{dy}{y'} = \frac{dy'}{0}$$

Integrating this system, we get

$$y' = C_1, \quad y = C_1 x + C_2$$

whence $C_2 = y - xy'$. Therefore, the general solution of equation (17) is

$$u(x, y, y') = \Phi(y', y - xy')$$

where Φ is an arbitrary differentiable function of its arguments. From this we have

$$F(x, y, z) = \alpha(x, y) + z\beta(x, y) + \int_0^z (z-t)\,\Phi(t, y-tx)\,dt \quad (18)$$

where $\alpha(x, y)$ and $\beta(x, y)$ are arbitrary functions of their arguments satisfying the relation

$$\frac{\partial \alpha}{\partial y} = \frac{\partial \beta}{\partial x}$$

It is evident from the solution that there exists an infinitude of variational problems for which equation (13) is Euler's equation.

Elementary cases of integrability of Euler's equation.

1°. F does not depend on y': $F = F(x, y)$.

In this case Euler's equation is of the form

$$F_y(x, y) = 0 \quad (19)$$

The solution of this finite equation does not contain any arbitrary elements and therefore, generally speaking, does not satisfy the boundary conditions $y(a) = A$, $y(b) = B$.

Only in exceptional cases where the curve (19) passes through the boundary points (a, A) and (b, B) does there exist a curve on which an extremum can be attained.

Example 8. Find the extremals of the functional

$$J[y(x)] = \int_0^{\pi/2} y(2x - y)\,dx, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Solution. Euler's equation is of the form $2x - 2y = 0$, that is, $y = x$. Since the boundary conditions are satisfied,

it follows that on the straight line $y = x$ the integral $\int_0^{\pi/2} y\,(2x-y)\,dx$ can attain an extremum. For other boundary conditions, for example, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$, the extremal $y = x$ does not pass through the boundary points $(0, 0)$ and $\left(\frac{\pi}{2}, 1\right)$ so that for these boundary conditions the variational problem does not have a solution.

2°. F is linearly dependent on y', that is,

$$F(x, y, y') = M(x, y) + N(x, y)\,y'$$

Euler's equation in this case has the form

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 \tag{20}$$

This equation, as in the case of 1°, is a finite and not a differential equation. The curve defined by the equation $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ does not, generally speaking, satisfy the boundary conditions and, hence, the variational problem does not, as a rule, have any solution in the class of continuous functions. If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \equiv 0$ in some domain D of the xy-plane, then the expression $F(x, y, y') = M(x, y)\,dx + N(x, y)\,dy$ is a total differential and the functional

$$J[y(x)] = \int_a^b F(x, y, y')\,dx = \int_{(a, A)}^{(b, B)} (M\,dx + N\,dy)$$

does not depend on the path of integration: the value of the functional $J[y(x)]$ is the same on admissible curves. The variational problem becomes meaningless.

Example 9. Investigate the following functional for an extremum:

$$J[y(x)] = \int_a^b (y^2 + 2xyy')\,dx,\ y(a) = A,\ y(b) = B$$

Solution. Here, F is linearly dependent on y'. We have

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y, \quad \text{and} \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \equiv 0$$

Hence, the integrand $(y^2 + 2xyy')\,dx$ is a total differential. Consequently, the integral does not depend on the path of integration:

$$J[y(x)] = \int_a^b (y^2\,dx + 2xy\,dy) = \int_{(a,\,A)}^{(b,\,B)} d(xy^2)$$

$$= xy^2 \Big|_{x=a}^{x=b} = bB^2 - aA^2$$

no matter along what curve $y(x)$ passing through the points (a, A) and (b, B) we integrate. The variational problem is meaningless.

3°. F depends only on y', that is, $F = F(y')$.

Euler's equation is of the form

$$F_{y'y'}y'' = 0 \tag{21}$$

In this case, the extremals are all possible straight lines

$$y = C_1x + C_2$$

where C_1 and C_2 are arbitrary constants.

Example 10. Find the extremals of the functional

$$J[y(x)] = \int_a^b \sqrt{1 + y'^2(x)}\,dx, \quad y(a) = A, \quad y(b) = B$$

This functional defines the length of the curve joining the points (a, A) and (b, B). Geometrically, the problem reduces to finding the shortest line connecting the given points.

Solution. Euler's equation has the form $y''(x) = 0$. The general solution is

$$y(x) = C_1x + C_2$$

The extremal that satisfies the boundary conditions $y(a) = A$ and $y(b) = B$ is obviously a straight line passing through the points (a, A) and (b, B):

$$y = \frac{B-A}{b-a}(x-a) + A$$

4°. F does not depend on y, that is, $F = F(x, y')$.

In this case the Euler equation $\frac{d}{dx} F_{y'}(x, y') = 0$, whence

$$F_{y'}(x, y') = C_1 \tag{22}$$

where C_1 is an arbitrary constant.

Equation (22) is a first-order differential equation. Integrating it, we find the extremals of the problem.

Example 11. From among the curves connecting the points A (1, 3) and B (2, 5), find the curve on which an extremum of the functional

$$J[y(x)] = \int_1^2 y'(x)(1 + x^2 y'(x))\, dx$$

can be attained.

Solution. Since F is not dependent on y, Euler's equation is of the form $\frac{d}{dx} F_{y'}(x, y') = 0$ or $\frac{d}{dx}(1 + 2x^2 y') = 0$, whence

$$1 + 2x^2 y' = C_1$$

Then $y' = \frac{C_1 - 1}{2x^2}$ so that $y(x) = \frac{C_1^*}{x} + C_2$, where $C_1^* = \frac{1 - C_1}{2}$.

Thus, the extremals are a family of hyperbolas. Let us isolate the extremal passing through the given points. To determine the constants C_1^* and C_2 we form the system

$$\left.\begin{aligned} 3 &= C_1^* + C_2, \\ 5 &= \frac{C_1^*}{2} + C_2 \end{aligned}\right\}$$

whence $C_1^* = -4$, $C_2 = 7$. The sought-for extremal is $y(x) = 7 - \frac{4}{x}$.

5°. F does not depend explicitly on x; that is, $F = F(y, y')$. In this case, Euler's equation assumes the form

$$F_y - F_{yy'} \cdot y' - F_{y'y'} \cdot y'' = 0$$

Multiplying both parts of this equation by y', we get an exact derivative in the left member, that is, $\frac{d}{dx}(F - y' \times \times F_{y'}) = 0$, whence

$$F - y' \cdot F_{y'} = C_1 \tag{23}$$

where C_1 is an arbitrary constant. This equation can be integrated by solving for y' and separating variables or by introducing a parameter.

Example 12. (Least resistance to a gas flow.) To determine the shape of a solid moving in a flow of gas with least

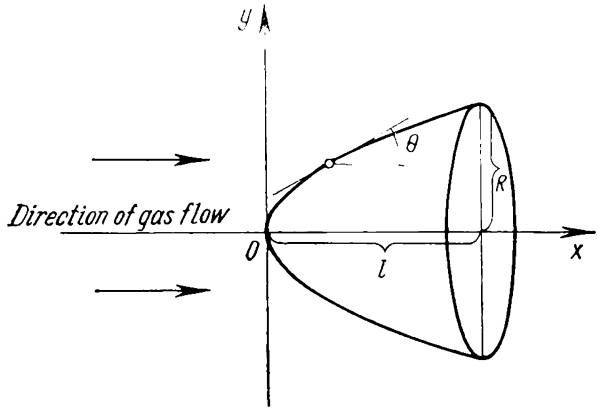

Fig. 4

resistance. For the sake of simplicity, we will consider a solid of revolution (Fig. 4).

Solution. Assuming that the gas density is sufficiently small and the molecules are mirror reflected from the surface of the solid, for the normal component of the pressure we will have the following expression:

$$p = 2\rho v^2 \sin^2 \theta \tag{24}$$

Here, ρ is the density of the gas, v the velocity of the gas relative to the solid, θ the angle between the velocity and its tangential component. The pressure is perpendicular to the surface so that it is possible to write down the force component along the x-axis acting on a ring of width

$(1 + y'^2)^{1/2}\,dx$ and radius $y\,(x)$ in the form

$$dF = 2\rho v^2 \sin^2\theta\,[2\pi y\,(1 + y'^2)^{1/2}]\sin\theta\,dx \qquad (25)$$

The total force acting in the positive x-direction is

$$F = \int_0^l 4\pi\rho v^2 \sin^3\theta y\,(1 + y'^2)^{1/2}\,dx \qquad (26)$$

In order to simplify the problem, let us assume that

$$\sin\theta = \frac{y'}{(1+y'^2)^{1/2}} \approx y'$$

Then the force of resistance will be

$$F = 4\pi\rho v^2 \int_0^l y'^3 y\,dx \qquad (27)$$

The problem consists in finding a function $y\,(x)$ for which F assumes the smallest possible value, and

$$y\,(0) = 0, \quad y\,(l) = R \qquad (28)$$

Euler's equation for the functional (27) is of the form

$$y'^3 - 3\frac{d}{dx}(yy'^2) = 0 \qquad (29)$$

The particular solution $y = 0$ of this equation is unacceptable due to the boundary conditions (28). Equation (29) may be rewritten in the form

$$y'^3 + 3yy'y'' = 0 \qquad (30)$$

Multiplying both members of (30) by y', we notice that the left member is $(y'^3 y)'$. Integrating, we get

$$y'^3 y = \widetilde{C}_1^3$$

whence

$$y' = \frac{\widetilde{C}_1}{\sqrt[3]{y}} \quad \text{and} \quad y = (C_1 x + C_2)^{3/4} \qquad (31)$$

Using the boundary conditions (28), we get

$$C_1 = \frac{R^{4/3}}{l}, \quad C_2 = 0$$

whence

$$y = R\left(\frac{x}{l}\right)^{3/4}$$

Thus, the contour with given endpoints for which the resistance of the solid is minimal is a parabola of degree 3/4.

Example 13. Find the extremal of the functional

$$J[y(x)] = \int_a^b \frac{\sqrt{1+y'^2}}{y}\,dx$$

which extremal passes through the given points (a, A) and (b, B) lying in the upper half-plane.

Solution. Since the integrand does not contain x explicitly, Euler's equation yields, by (23),

$$\frac{\sqrt{1+y'^2}}{y} - y'\frac{y'}{y\sqrt{1+y'^2}} = C_1$$

Simplifying, we get $y\sqrt{1+y'^2} = \widetilde{C}_1$, where $\widetilde{C}_1 = \frac{1}{C_1}$. Integrating the latter equation, we find $(x + C_2)^2 + y^2 = \widetilde{C}_1^2$, which is a family of circles centred on the x-axis. The desired extremal is that one which passes through the given points. The problem has a unique solution since one and only one semicircle centred on the x-axis passes through any two points lying in the upper half-plane.

Note. According to Fermat's principle, the path of a light ray moving in an inhomogeneous two-dimensional medium with velocity $v(x, y)$ is an extremal of the functional

$$J[y] = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{v(x, y)}\,dx$$

If the velocity of light, v, is proportional to y, then, as may be seen from the example examined above, the light rays are arcs of circles whose centres lie on the x-axis.

Let there be given a curve q. The *optical path length* q is the time $T(q)$ during which the curve is traversed with the velocity of light, $v(x, y)$.

We will consider the upper half-plane $y > 0$ as an optical medium in which the velocity of light at every point is equal to the ordinate of that point, $v = y$. As we have already seen, the light rays in this medium will be semicircles with centres on the x-axis. It may be shown that the part AD of the semicircle q, one of the ends of which lies on the x-axis, has an infinite optical path length (Fig. 5). We will

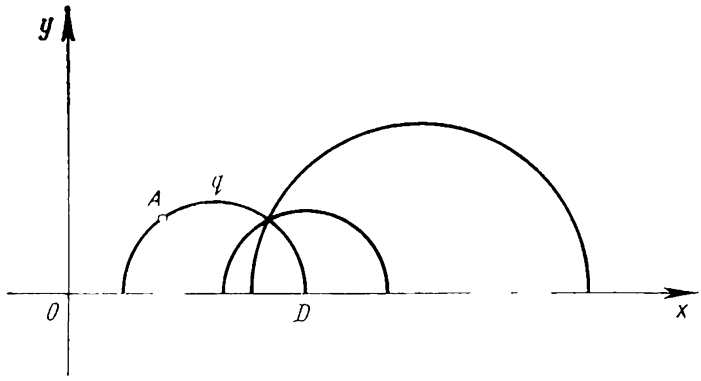

Fig. 5

therefore call points on the x-axis *infinite points*. We will consider the semicircles with centres on the x-axis to be *straight lines*, the optical path lengths of the arcs of such semicircles to be their *lengths*, and the *angles* between the tangents to the semicircles at their intersections to be the angles between such straight lines. We obtain a flat geometry in which many of the propositions of ordinary geometry are preserved. For example, one and only one *straight line* can be drawn through two points (only one semicircle centred on the x-axis can be drawn through two points on the semicircle). Two *straight lines* will be said to be *parallel* if they have a common point at infinity (that is, two semicircles tangent to each other at a point B lying on the x-axis). Then it is possible to draw, through the given point A not lying on the *straight line* q, two *straight lines* q_1 and q_2 parallel to q. The *straight lines* that pass through point A and lie in the vertical angles I and III intersect the straight line q. The straight lines lying in the angles II and IV do not intersect q.

We have obtained what is called the Poincaré model of Lobachevskian geometry in the plane (Fig. 6).

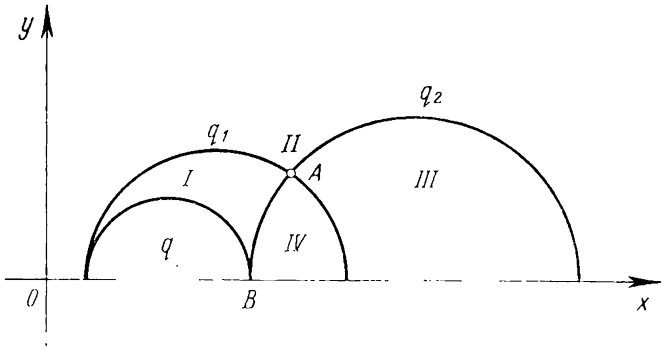

Fig. 6

Find the extremals of the following functionals.

81. $J[y(x)] = \int_a^b [2xy + (x^2 + e^y)\, y']\, dx;$

$$y(a) = A, \quad y(b) = B.$$

82. $J[y(x)] = \int_0^1 (e^y + xy')\, dx; \; y(0) = 0, \; y(1) = \alpha.$

83. $J[y(x)] = \int_0^{\pi/4} (y'^2 - y^2)\, dx; \; y(0) = 1, \; y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$

84. $J[y(x)] = \int_0^{\pi} (y'^2 - y^2)\, dx; \; y(0) = 1, \; y(\pi) = -1.$

85. $J[y(x)] = \int_0^1 (x + y'^2)\, dx; \; y(0) = 1, \; y(1) = 2.$

86. $J[y(x)] = \int_0^1 (y^2 + y'^2)\, dx; \; y(0) = 0, \; y(1) = 1.$

87. $J[y(x)] = \int_0^1 (y'^2 + 4y^2)\, dx; \; y(0) = e^2, \; y(1) = 1.$

88. $J[y(x)] = \int_0^1 (2e^y - y^2)\,dx;\ y(0) = 1,\ y(1) = e.$

89. $J[y(x)] = \int_a^b (xy' + y'^2)\,dx.$

90. $J[y(x)] = \int_a^b \left(y + \frac{y^3}{3}\right) dx.$

91. Show that the linear functional

$$J[y(x)] = \int_a^b [p(x)\,y' + q(x)\,y + r(x)]\,dx$$

where $p(x) \in C_1[a, b]$, $q(x) \in C[a, b]$, $r(x) \in C[a, b]$, does not have any extrema.

92. Given the functional

$$J[y(x)] = \int_a^b F(x, y, y')\,dx$$

and the boundary conditions $y(a) = A$, $y(b) = B$.

Show that if to the integrand $F(x, y, y')\,dx$ we add the total differential of any function $u = u(x, y)$, then Euler's equation will remain unchanged.

93. $J[y(x)] = \int_a^b (y^2 + y'^2 + 2ye^x)\,dx.$

94. $J[y(x)] = \int_0^{\pi/2} (y^2 - y'^2 - 8y\cosh x)\,dx;$

$$y(0) = 2,\ y\left(\frac{\pi}{2}\right) = 2\cosh\frac{\pi}{2}.$$

95. Find the extremals of the functional

$$J[y(x)] = \int_a^b x^n (y')^2\,dx$$

and show that for $n \geqslant 1$ two points lying on different sides of the y-axis cannot be joined by an extremal.

Variational problems in parametric form. In a number of problems it is more convenient and sometimes simply necessary to make use of a parametric representation of the lines,

$$\left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\}, \quad t_0 \leqslant t \leqslant t_1$$

where the functions $\varphi(t)$ and $\psi(t)$ are continuous and have piecewise continuous derivatives, and $\dot{\varphi}^2(t) + \dot{\psi}^2(t) \neq 0$.

Suppose we have the functional

$$J_C = \int_C F(t, x, y, \dot{x}, \dot{y})\, dt = \int_{t_0}^{t_1} F(t, x, y, \dot{x}, \dot{y})\, dt \qquad (32)$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$.

In order that the values of the functional (32) should depend on the line and not on its parametrization, which can be accomplished in various ways, it is necessary and sufficient that the integrand not contain explicitly the parameter t and that it be positive homogeneous of first degree in the arguments $\dot{x}$, $\dot{y}$:

$$F(x, y, k\dot{x}, k\dot{y}) = kF(x, y, \dot{x}, \dot{y}), \quad k > 0 \qquad (33)$$

For example, in the functional

$$J_C = \int_C x\, dy - y\, dx$$

the integrand is positive homogeneous of first degree. Indeed, here

$$F(x, y, \dot{x}, \dot{y}) = x\dot{y} - y\dot{x}$$

and clearly

$$F(x, y, k\dot{x}\ k\dot{y}) = kF(x, y, \dot{x}, \dot{y})$$

If the line $\widetilde{C}$:

$$\left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\}, \quad t_0 \leqslant t \leqslant t_1$$

extremizes the functional J_C in the class of lines C connecting the given points (x_0, y_0) and (x_1, y_1), then the functions $\varphi(t)$ and $\psi(t)$ satisfy the Euler equations

$$\left.\begin{aligned} F_x - \frac{d}{dt}(F_{\dot{x}}) = 0, \\ F_y - \frac{d}{dt}(F_{\dot{y}}) = 0 \end{aligned}\right\} \tag{34}$$

One of the equations of (34) is a consequence of the other.

The Weierstrassian form of Euler's equation is

$$\frac{1}{r} = \frac{F_{\dot{x}\dot{y}} - F_{\dot{y}\dot{x}}}{F_1 \cdot (\dot{x}^2 + \dot{y}^2)^{3/2}} \tag{35}$$

Here, r is the radius of curvature of the extremal, F_1 is the common value of the ratios

$$F_1 = \frac{F_{\dot{x}\dot{x}}}{\dot{y}^2} = \frac{F_{\dot{y}\dot{y}}}{\dot{x}^2} = \frac{F_{\dot{x}\dot{y}}}{-\dot{x}\dot{y}}$$

Example 14. Find the extremals of the functional

$$J_C = \int_{(0,\,0)}^{(x_1,\,y_1)} y^2 y'^2 \, dx$$

Solution. Since extremals are possible that are intersected by straight lines parallel to the y-axis in more than one point, let us consider the problem in parametric form.

Assuming $x = x(t)$, $y = y(t)$, we find the integrand is of the form $y^2 \cdot \frac{\dot{y}^2}{\dot{x}^2}\dot{x}$ and is a positive homogeneous function of the first degree in $\dot{x}$ and $\dot{y}$.

The first equation of (34) assumes the form

$$\frac{d}{dt}\left(y^2 \frac{\dot{y}^2}{\dot{x}}\right) = 0$$

whence

$$y^2 \left(\frac{dy}{dx}\right)^2 = C_1^2$$

Integrating the latter equation, we find

$$y^2 = 2C_1x + C_2$$

From the condition that the extremal passes through the origin, we find that $C_2 = 0$. The second boundary condition yields $C_1 = \frac{y_1^2}{2x_1}$ so that, finally,

$$y^2 = \frac{y_1^2}{x_1}\, x$$

Example 15. Find the extremals of the functional

$$J_C = \int_{t_0}^{t_1} \left[\sqrt{\dot{x}^2 + \dot{y}^2} + a^2(x\dot{y} - y\dot{x})\right] dt$$

Solution. Putting

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} + a^2(x\dot{y} - y\dot{x})$$

we see that the function F is positive homogeneous of the first degree in $\dot{x}$ and $\dot{y}$.

Taking advantage of the Weierstrassian form of Euler's equation, we have

$$F_{\dot{x}\dot{y}} = a^2, \quad F_{\dot{y}\dot{x}} = -a^2, \quad F_1 = \frac{F_{\dot{x}\dot{x}}}{\dot{y}^2} = \frac{1}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

For this reason, equation (35) assumes in this case the form

$$\frac{1}{r} = 2a^2$$

Thus, the curvature $\frac{1}{r}$ of the extremal is constant. Hence, the extremals are arcs of circles, in particular, full circles if

$$\left.\begin{aligned} x(t_0) &= x(t_1), \\ y(t_0) &= y(t_1) \end{aligned}\right\}$$

Find the extremals of the following functionals.

96. $J_C = \int_{(0,\,0)}^{(x_1,\,y_1)} \frac{\dot{y}^2 - y^2\dot{x}^2}{\dot{x}}\, dt.$

97. $J_C = \int\limits_{(0,\,0)}^{(1,\,2)} \frac{\dot{y}^2 - 3e^{\dot{y}/\dot{x}}\dot{x}^2}{\dot{x}}\,dt.$

98. $J_C = \int\limits_{(-1,\,0)}^{(1,\,0)} \left(K\cdot\sqrt{\dot{x}^2+\dot{y}^2} - \dot{x}y\right)dt,$

$K > 0$ — constant.

5. GENERALIZATIONS OF THE ELEMENTARY PROBLEM OF THE CALCULUS OF VARIATIONS

1°. Functionals dependent on higher derivatives. Suppose we have the functional

$$J[y(x)] = \int\limits_{x_0}^{x_1} F(x,\ y(x),\ y'(x),\ \ldots,\ y^{(n)}(x))\,dx \qquad (1)$$

where F is an $n+2$ times differentiable function with respect to all arguments, $y(x) \in C_n[x_0,\ x_1]$, and the boundary conditions have the form

$$\left.\begin{aligned} &y(x_0) = y_0,\ y'(x_0) = y_0',\ \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)},\\ &y(x_1) = y_1,\ y'(x_1) = y_1',\ \ldots,\ y^{(n-1)}(x_1) = y_1^{(n-1)} \end{aligned}\right\} \qquad (2)$$

The extremals of the functional (1), given conditions (2), are the integral curves of the Euler-Poisson equation:

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \ldots + (-1)^n\frac{d^n}{dx^n}F_{y^{(n)}} = 0 \qquad (3)$$

Example 1. Find the extremal of the functional

$$J[y(x)] = \int\limits_0^1 (360x^2y - y''^2)\,dx$$

$$y(0) = 0,\ y'(0) = 1,\ y(1) = 0,\ y'(1) = 2.5$$

Solution. The Euler-Poisson equation is of the form

$$360x^2 + \frac{d^2}{dx^2}(-2y'') = 0 \text{ or } y^{IV}(x) = 180x^2$$

and the general solution is

$$y(x) = \frac{1}{2}x^6 + C_1x^3 + C_2x^2 + C_3x + C_4$$

Using the boundary conditions, we get

$$C_1 = \frac{3}{2}, \quad C_2 = -3, \quad C_3 = 1, \quad C_4 = 0$$

The desired extremal is

$$y(x) = \frac{1}{2}x^6 + \frac{3}{2}x^3 - 3x^2 + x$$

Let us consider a problem in which not all the conditions (2) are given on the boundary, but a smaller number of conditions, so that in the general solution of Euler's equation there remain certain free constants after using the boundary conditions. To solve such a problem it is necessary to find a variation of the functional (1), transform it taking into account the given boundary conditions and, equating the variation to zero, obtain supplementary conditions on the boundary.

Example 2. Find the curve $y = y(x)$ that extremizes the functional

$$J[y] = \frac{1}{2}\int_a^b (y'')^2\,dx \tag{4}$$

subject to the conditions

$$y(a) = 0, \quad y(b) = 0 \tag{5}$$

Solution. The Euler-Poisson equation has the form

$$y^{(IV)}(x) = 0$$

The general solution is

$$y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 \tag{6}$$

contains four arbitrary constants C_i $(i = 1, 2, 3, 4)$, and the given boundary conditions (5) are insufficient for their determination. Therefore, according to the foregoing, we find a variation of the functional (4). We have

$$\delta J[y] = \int_a^b y''(\delta y'')\,dx \tag{7}$$

Integrating (7) twice by parts, we get

$$\delta J[y] = y''(x)\,\delta y'(x)\Big|_a^b - \int_a^b y'''(x)\,\delta y'(x)\,dx$$

$$= y''(x)\,\delta y'(x)\Big|_a^b - y'''(x)\,\delta y(x)\Big|_a^b + \int_a^b y^{(IV)}(x)\,\delta y(x)\,dx \qquad (8)$$

Expression (8) should vanish on the extremal $y(x)$ of the functional (4). From the arbitrariness of the function $\delta y(x)$ it follows that $y^{(IV)}(x) = 0$. This is the Euler-Poisson function for the functional (4). But if the integral in the right member of (8) vanishes, then the *boundary expression*

$$[y''(x)\,\delta y'(x) - y'''(x)\,\delta y(x)]\Big|_a^b$$

must also vanish identically.

Since $\delta y(a) = \delta y(b) = 0$ (the ends are fixed), we find that it must be true that

$$y''(b)\,\delta y'(b) - y''(a)\,\delta y'(a) = 0$$

Since the quantities $\delta y'(b)$ and $\delta y'(a)$ are arbitrary, we necessarily obtain

$$y''(a) = 0,\quad y''(b) = 0 \qquad (9)$$

The conditions (9), together with conditions (5), unambiguously isolate the extremal from the family (6): $y(x) \equiv 0$.

2°. Functionals dependent on m functions. For a functional that is dependent on m functions $y_1(x), \ldots, y_m(x)$,

$$J[y_1, y_2, \ldots, y_m] = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_m, y_1', y_2', \ldots, y_m')\,dx \qquad (10)$$

given boundary conditions of the form

$$y_k(x_0) = y_k^{(0)},\quad y_k(x_1) = y_k^{(1)}\quad (k = 1, 2, \ldots, m) \qquad (11)$$

the extremals are found from the following system of second-order differential equations:

$$F_{y_k} - \frac{d}{dx} F_{y_k'} = 0 \quad (k = 1, 2, \ldots, m) \tag{12}$$

called a system of Euler's equations.

Example 3. Find the extremals of the functional

$$J[y(x), z(x)] = \int_1^2 (y'^2 + z^2 + z'^2)\, dx$$

for the boundary conditions

$$y(1) = 1, \quad y(2) = 2, \quad z(1) = 0, \quad z(2) = 1$$

Solution. In this case the system of equations (12) is of the following form:

$$\left.\begin{aligned} y'' &= 0, \\ z - z'' &= 0 \end{aligned}\right\}$$

Solving this system, we find

$$y = C_1 x + C_2, \quad z = C_3 e^x + C_4 e^{-x}$$

By virtue of the boundary conditions we have

$$C_1 = 1, \; C_2 = 0, \; C_3 = \frac{1}{e^2 - 1}, \; C_4 = -\frac{e^2}{e^2 - 1}$$

so that the desired extremal,

$$\left.\begin{aligned} y &= x, \\ z &= \frac{\sinh(x-1)}{\sinh 1} \end{aligned}\right\}$$

is a space curve which is the intersection of two cylindrical surfaces.

Example 4. Find the extremals of the functional

$$J[y(x), z(x)] = \int_0^{\pi} (2yz - 2y^2 + y'^2 - z'^2)\, dx$$

if

$$y(0) = 0, \quad y(\pi) = 1, \quad z(0) = 0, \quad z(\pi) = -1$$

Solution. The system of equations (12) has the form

$$\left.\begin{aligned} y'' + 2y - z = 0, \\ z'' + y = 0 \end{aligned}\right\}$$

whence, eliminating the function z, we get

$$y^{(IV)} + 2y'' + y = 0$$

The general solution of this equation is

$$y(x) = C_1 \cos x + C_2 \sin x + x(C_3 \cos x + C_4 \sin x)$$

By virtue of the boundary conditions $y(0) = 0$, $y(\pi) = 1$ we get $C_1 = 0$, $C_3 = -\frac{1}{\pi}$ and, hence,

$$y(x) = C_2 \sin x + C_4 x \sin x - \frac{x}{\pi} \cos x$$

We find the function z from the condition that $z = y'' + 2y$ and we have

$$z = C_2 \sin x + C_4 (2 \cos x + x \sin x) + \frac{1}{\pi} (2 \sin x - x \cos x)$$

The constants C_2 and C_4 are found from the boundary conditions $z(0) = 0$, $z(\pi) = -1$; this yields $C_4 = 0$ and C_2 is arbitrary. Then

$$z = C_2 \sin x + \frac{1}{\pi} (2 \sin x - x \cos x)$$

The family of extremals is

$$\left.\begin{aligned} y = C_2 \sin x - \frac{x}{\pi} \cos x, \\ z = C_2 \sin x + \frac{1}{\pi} (2 \sin x - x \cos x) \end{aligned}\right\}$$

where C_2 is an arbitrary constant.

3°. Functionals dependent on functions of several independent variables. Let us consider a functional of the form

$$J[z(x, y)] = \iint_D F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\, dy \qquad (13)$$

where F is a three-times differentiable function of its arguments, and let us assume that a function $z = z(x, y)$ is being sought that is continuous together with its derivatives up to order two inclusive in the domain D and assumes on the boundary Γ of D given values and yields an extremum of the functional (13).

If an extremum of the functional (13) is realized on the surface $z = z(x, y)$, then the function $z = z(x, y)$ satisfies the Euler-Ostrogradsky equation

$$F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\} = 0 \qquad (14)$$

where $\frac{\partial}{\partial x}\{F_p\}$ and $\frac{\partial}{\partial y}\{F_q\}$ are total partial derivatives with respect to x and y, respectively:

$$\frac{\partial}{\partial x}\{F_p\} = F_{px} + F_{pz}\frac{\partial z}{\partial x} + F_{pp}\frac{\partial p}{\partial x} + F_{pq}\frac{\partial q}{\partial x}, \qquad (15)$$

$$\frac{\partial}{\partial y}\{F_q\} = F_{qy} + F_{qz}\frac{\partial z}{\partial y} + F_{qp}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y} \qquad (16)$$

Here, we put $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ for the sake of brevity.

Equation (14) is a necessary condition for an extremum of the functional (13). It is a second-order partial differential equation; we seek the solution $z = z(x, y)$ that takes on the specified values on the boundary Γ.

Example 5. Write down the Euler-Ostrogradsky equation for the functional

$$J[z(x, y)] = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2\right] dx\, dy$$

Solution. We have $F(x, y, z, p, q) = p^2 - q^2$ so that, by (14), we get $-\frac{\partial}{\partial x}(2p) - \frac{\partial}{\partial y}(-2q) = 0$ or

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

For the functional

$$J[z(x_1, x_2, \ldots, x_n)] = \int\int \ldots \int_D F(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n)\, dx_1\, dx_2 \ldots dx_n \qquad (17)$$

where $p_k = \frac{\partial z}{\partial x_k}$ $(k = 1, 2, \ldots, n)$, the necessary condition of an extremum is expressed by the following Euler-Ostrogradsky equation:

$$F_z - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \{F_{p_i}\} = 0 \tag{18}$$

or, expanded,

$$F_z - \sum_{i=1}^{n} \left(F_{x_i p_i} + F_{z p_i} \cdot p_i + \sum_{j=1}^{n} F_{p_i p_j} \frac{\partial p_j}{\partial x_i} \right) = 0 \tag{19}$$

The solution of this equation—the function $z = z(x_1, x_2, \ldots, x_n)$—on the boundary Γ of the n-dimensional domain D must satisfy the given boundary conditions.

Example 6. Find the conditions for which the function $z(x_1, x_2, \ldots, x_n)$, which assumes specified values on the boundary Γ of the domain Q, minimizes the Dirichlet integral

$$D(z) = \int\int_{Q} \ldots \int \sum_{i=1}^{n} \left(\frac{\partial z}{\partial x_i} \right)^2 dx_1 \, dx_2 \ldots dx_n$$

Solution. In this case, $F = \sum_{i=1}^{n} \left(\frac{\partial z}{\partial x_i} \right)^2$, that is, F does not explicitly depend on $x_1, x_2, \ldots, x_n, z$. Hence,

$$F_z = F_{z p_i} = F_{x_i p_i} = 0, \quad F_{p_i p_j} = \begin{cases} 2 \text{ when } i = j, \\ 0 \text{ when } i \neq j \end{cases}$$

and by (19) we get

$$\sum_{j=1}^{n} \frac{\partial^2 z}{\partial x_j^2} = 0 \text{ or } \Delta z = 0$$

(an n-dimensional Laplace equation).

Note. If derivatives of the function $z(x, y)$ up to order n appear under the integral sign, then the Euler-Ostrogradsky

equation becomes

$$F_z - \frac{\partial}{\partial x}\{F_{z_x}\} - \frac{\partial}{\partial y}\{F_{z_y}\} + \frac{\partial^2}{\partial x^2}\{F_{z_{xx}}\} + 2\frac{\partial^2}{\partial x\,\partial y}\{F_{z_{xy}}\} + \frac{\partial^2}{\partial y^2}\{F_{z_{yy}}\} - \ldots + (-1)^n \frac{\partial^n}{\partial y^n}\{F_{z\underbrace{yy\ldots y}_{n}}\} = 0 \qquad (20)$$

Example 7. Write down the Euler-Ostrogradsky equation for the functional

$$J[z(x, y)] = \iint_D \left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 - 2zf(x, y)\right] dx\,dy$$

Solution. We have

$$F = \left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 - 2zf(x, y)$$

According to (20), we find

$$-2f(x, y) + \frac{\partial^2}{\partial x^2}\left(2\frac{\partial^2 z}{\partial x^2}\right) + \frac{\partial^2}{\partial y^2}\left(2\frac{\partial^2 z}{\partial y^2}\right) + 2\frac{\partial^2}{\partial x\,\partial y}\left(2\frac{\partial^2 z}{\partial x\,\partial y}\right) = 0$$

or

$$\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2\,\partial y^2} + \frac{\partial^4 z}{\partial y^4} = f(x, y)$$

This latter equation is briefly written thus:

$$\Delta\Delta z = f(x, y)$$

Find the extremals of the following functionals.

99. $J[y(x)] = \int_0^1 (y^2 + 2y'^2 + y''^2)\,dx;$

$y(0) = 0, \quad y(1) = 0, \quad y'(0) = 1, \quad y'(1) = -\sinh 1.$

100. $J[y(x)] = \int_{-1}^{0} (240y - y''^2)\,dx;$

$$y(-1) = 1, \quad y(0) = 0, \quad y'(-1) = -4.5,$$
$$y'(0) = 0, \quad y''(-1) = 16, \quad y''(0) = 0.$$

101. $J[y(x)] = \int_{a}^{b} (y + y'')\,dx;$

$$y(a) = y_0, \quad y(b) = y_1, \quad y'(a) = y_0', \quad y'(b) = y_1'.$$

102. $J[y(x)] = \int_{a}^{b} (y'^2 + yy'')\,dx;$

$$y(a) = A_1, \quad y'(a) = A_2, \quad y(b) = B_1, \quad y'(b) = B_2.$$

103. $J[y(x)] = \int_{0}^{1} (y'^2 + y''^2)\,dx;$

$$y(0) = 0, \quad y(1) = \sinh 1, \quad y'(0) = 1, \quad y'(1) = \cosh 1.$$

104. Find the extremal of the functional

$$J[y] = \frac{1}{2} \int_{0}^{1} (y'')^2\,dx$$

subject to the conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y'(1) = 1.$$

105. $J[y(x), z(x)] = \int_{0}^{\pi/4} (2z - 4y^2 + y'^2 - z'^2)\,dx;$

$$y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = 1, \quad z(0) = 0, \quad z\left(\frac{\pi}{4}\right) = 1.$$

106. $J[y(x), z(x)] = \int_{-1}^{1} \left(2xy - y'^2 + \frac{z'^3}{3}\right) dx;$

$$y(1) = 0, \quad y(-1) = 2, \quad z(1) = 1, \quad z(-1) = -1.$$

107. $J[y(x), z(x)] = \int_0^{\pi/2} (y'^2 + z'^2 - 2yz)\,dx;$

$$y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1, \quad z(0) = 0, \quad z\left(\frac{\pi}{2}\right) = 1.$$

108. $J[y(x), z(x)] = \int_0^1 (y'^2 + z'^2 + 2y)\,dx;$

$$y(0) = 1, \quad y(1) = \frac{3}{2}, \quad z(0) = 0, \quad z(1) = 1.$$

109. Show that the Euler equations for the functional

$$J[y, z] = \int_a^b F(x, y, y', z, z')\,dx$$

admit the following first integrals:

(1) $\frac{\partial F}{\partial y'} = C$ if F does not contain y;

(2) $F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = C$ if F does not contain x.

Write down the Euler-Ostrogradsky equations for the following functionals.

110. $J[z(x, y)] = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^4 + \left(\frac{\partial z}{\partial y}\right)^4 + 12zf(x,y)\right] dx\,dy.$

111. $J[z(x, y)] = \iint_D \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)^2 dx\,dy.$

112. $J[z(x_1, x_2, \ldots, x_n)]$

$$= \int\int \ldots \int_D \left[\sum_{j=1}^n a_j(x_1, \ldots, x_n)\left(\frac{\partial z}{\partial x_j}\right)^2 - c(x_1, \ldots, x_n)\,z^2 + 2zf(x_1, \ldots, x_n)\right] dx_1\,dx_2 \ldots dx_n.$$

113. Derive the differential equation for minimal surfaces.

114. Find the extremal of the functional

$$J[z(x, y)] = \int_0^1 \int_0^1 e^{zy} \sin z_y \, dx \, dy$$

given the conditions $z(x, 0) = 0$, $z(x, 1) = 1$.

6. INVARIANCE OF EULER'S EQUATION

If a functional

$$J[y] = \int_a^b F(x, y, y') \, dx$$

can be transformed by replacing the independent variable or by a simultaneous replacement of the desired function and the independent variable, then the extremals of the functional are, as before, found from the Euler equation set up for the transformed integrand. Therein lies the invariance of Euler's equation.

Let $x = x(u, v)$, $y = y(u, v)$, and

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$$

Then

$$\int F(x, y, y') \, dx$$

$$= \int F\left[x(u, v), \; y(u, v), \; \frac{y_u + y_v v'_u}{x_u + x_v v'_u}\right](x_u + x_v v'_u) \, du$$

$$= \int \Phi(u, v, v'_u) \, du$$

and the extremals of the original functional are determined from the Euler equation for the functional $\int \Phi(u, v, v'_u) \, du$:

$$\Phi_v - \frac{d}{du} \Phi_{v'} = 0$$

Example 1. Find the extremals of the functional

$$J[r] = \int_{\varphi_0}^{\varphi_1} \sqrt{r^2 + r'^2} \, d\varphi, \qquad \text{where} \qquad r = r(\varphi)$$

Solution. The Euler equation for this functional is

$$\frac{r}{\sqrt{r^2+r'^2}}-\frac{d}{d\varphi}\left(\frac{r'}{\sqrt{r^2+r'^2}}\right)=0$$

The change of variables $x=r\cos\varphi$, $y=r\sin\varphi$ yields

$$\sqrt{r^2+r'^2}\,d\varphi=\sqrt{1+y'^2}\,dx$$

and we arrive at a functional of the form

$$J[y]=\int_a^b \sqrt{1+y'^2}\,dx$$

for which the Euler equation is $y''=0$ so that

$$y=C_1x+C_2$$

Hence, the extremals of the original functional are given by the equations

$$r\sin\varphi=C_1r\cos\varphi+C_2$$

where C_1 and C_2 are arbitrary constants.

Example 2. Find the extremals of the functional

$$J[y]=\int_0^{\ln 2}(e^{-x}y'^2-e^xy^2)\,dx$$

Solution. Euler's equation for this functional is of the form

$$y''-y'+e^{2x}y=0$$

We make a change of variables:

$$\begin{cases} x=\ln u, \\ y=v \end{cases}$$

Then the original functional becomes

$$J_1[v]=\int_1^2 (e^{-\ln u}u^2v'^2-e^{\ln u}v^2)\frac{du}{u}=\int_1^2 (v'^2-v^2)\,du$$

and for it the Euler equation $v''+v=0$ is readily integrated:

$$v=C_1\cos u+C_2\sin u$$

Passing to the original coordinates x, y, we get the equation of the extremals in the form

$$y = C_1 \cos e^x + C_2 \sin e^x$$

115. Find the extremals of the functional

$$J = \int_{\varphi_0}^{\varphi_1} r \sin \varphi \sqrt{r^2 + r'^2}\, d\varphi .$$

116. Show that the extremals of the functional

$$J = \int_{\varphi_0}^{\varphi_1} f(r \sin \varphi) \sqrt{r^2 + r'^2}\, d\varphi$$

can always be found in quadratures.

117. Find the extremals of the functional

$$J = \int_a^b \sqrt{x^2 + y^2} \sqrt{1 + y'^2}\, dx$$

As in the case of one variable, the Euler-Ostrogradsky equation is invariant with respect to transformations of coordinates.

Example 3. Write down the Laplace equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad (1)$$

in polar coordinates.

Solution. Consider the functional

$$D[z(x, y)] = \iint_G (z_x^2 + z_y^2)\, dx\, dy$$

The Euler-Ostrogradsky equation for the functional is precisely equation (1). Let us pass from Cartesian coordinates (x, y) to polar coordinates (ρ, φ): $x = \rho \cos \varphi$, $y = \rho \sin \varphi$. We have

$$\frac{\partial \rho}{\partial x} = \cos \varphi, \quad \frac{\partial \rho}{\partial y} = \sin \varphi, \quad \frac{\partial \varphi}{\partial x} = -\frac{\sin \varphi}{\rho}, \quad \frac{\partial \varphi}{\partial y} = \frac{\cos \varphi}{\rho}$$

whence

$$D[z(\rho, \varphi)]$$
$$= \iint_G \left[\left(z_\rho \frac{\partial \rho}{\partial x} + z_\varphi \frac{\partial \varphi}{\partial x}\right)^2 + \left(z_\rho \frac{\partial \rho}{\partial y} + z_\varphi \frac{\partial \varphi}{\partial y}\right)^2\right] \rho \, d\rho \, d\varphi$$
$$= \iint_G \left(\rho z_\rho^2 + \frac{1}{\rho} z_\varphi^2\right) d\rho \, d\varphi$$

Forming the Euler-Ostrogradsky equation for the latter integral, we arrive at the Laplace equation in polar coordinates

$$\frac{1}{\rho} z_{\varphi\varphi} + \rho z_{\rho\rho} + z_\rho = 0$$

7. FIELD OF EXTREMALS

A family of curves $y = y(x, c)$ forms a *proper field in a given domain* D of the xy-plane if through every point (x, y) of the domain there passes one and only one curve of the family $y = y(x, c)$.

The gradient $\rho(x, y)$ of the tangent to a curve of the family $y = y(x, c)$ passing through the point (x, y) is called the *slope of the field at the point* (x, y).

A family of curves $y = y(x, c)$ forms a *central field* in a domain D of the xy-plane if these curves cover the entire domain D without self-intersections and issue from a single point (x_0, y_0) lying outside D. The point (x_0, y_0) is called the *centre of the pencil of curves*.

Example 1. Inside the circle $x^2 + y^2 \leqslant 1$, the family of curves $y = Ce^x$, where C is an arbitrary constant, in particular, $C = 0$, forms a proper field, since these curves do not intersect anywhere and through every point (x, y) of the circle there passes one and only one curve of the family (Fig. 7). The slope of the field at an arbitrary point (x, y) is equal to

$$\rho(x, y) = Ce^x = y$$

Example 2. The family of parabolas $y = (x + C)^2$ inside the circle $x^2 + y^2 \leqslant 1$ does not form a proper field since different curves of the family intersect inside the circle and do not cover the entire domain (Fig. 8).

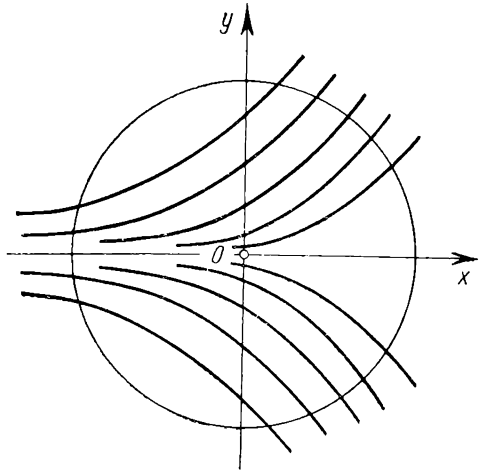

Fig. 7

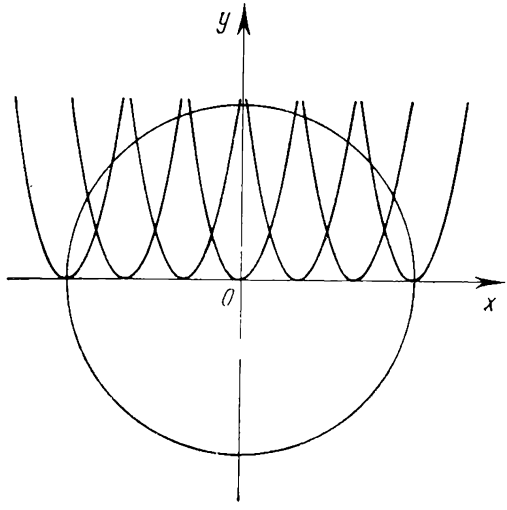

Fig. 8

Example 3. The family of curves $y = Cx$ forms a central field in the domain $x > 0$.

Do the following families of curves form fields (proper or central) in the indicated domains?

118. $y = C \cdot \tan x$; $0 \leqslant x \leqslant \frac{\pi}{4}$, $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

119. $y = C \cdot \cos x$;

(a) $|x| < \frac{\pi}{4}$, (b) $\frac{\pi}{2} < x \leqslant \pi$, (c) $|x| \leqslant \pi$.

120. $y = (x - C)^3$; $\frac{x^2}{4} + \frac{y^2}{9} \leqslant 1$.

121. $y = C(x^2 - 2x)$;

(a) $0 \leqslant x < 1$, (b) $-1 \leqslant x \leqslant 3$, (c) $\frac{1}{2} \leqslant x \leqslant \frac{3}{2}$.

122. $y = C \cdot \sin\left(x - \frac{\pi}{4}\right)$;

(a) $\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{2}$, (b) $\frac{\pi}{3} \leqslant x \leqslant \pi$, (c) $\frac{\pi}{8} \leqslant x \leqslant 2\pi$.

123. $y = e^{x+C}$; $x^2 + y^2 \leqslant 1$.

If a field (proper or central) is formed by a family of extremals of some variational problem, then it is called a *field of extremals* (*extremal field*).

Example 4. Let us consider the functional

$$J[y] = \int_0^1 y'^2 \, dx$$

Its extremals are the straight lines $y = C_1 x + C_2$. The family of extremals $y = C_2$ forms a proper field, the family of extremals $y = C_1 x$ forms a central field centred at the coordinate origin.

124. Indicate the proper and central field of extremals for the functional

$$J[y] = \int_0^a (y'^2 + y^2) \, dx, \; a > 0$$

125. Do the same for the functional

$$J[y] = \int_0^{\pi/4} (y'^2 - y^2 + x^2 + 4)\, dx$$

Let the curve $y = y(x)$ be an extremal of the functional

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

which extremal passes through the points $A(x_0, y_0)$ and $B(x_1, y_1)$.

We say that the extremal $y = y(x)$ is *included in the proper field of extremals* if there is a family of extremals $y = y(x, C)$ that forms a field containing the extremal $y = y(x)$ for some value $C = C_0$; this extremal $y = y(x)$ does not lie on the boundary of the domain D in which the family $y = y(x, C)$ forms a field.

If a pencil of extremals centred at the point (x_0, y_0) in the neighbourhood of the extremal $y = y(x)$ passing through the same point forms a field, then we say that a *central field has been found* that includes the given extremal $y = y(x)$. For the parameter of the family $y = y(x, C)$ we take the slope of the tangent to the curves of the pencil at the point (x_0, y_0).

Example 5. Let us consider the elementary variational problem for the functional

$$J[y] = \int_0^2 (y'^3 + \sin^2 x)\, dx$$

(a) Let $y(0) = 1$, $y(2) = 1$. The family of extremals of the given functional is determined by the equation $y = C_1 x + C_2$. The given boundary conditions are satisfied by the extremal $y = 1$. This extremal is included in the proper field of extremals $y = C_2$, where C_2 is an arbitrary constant.

(b) Let $y(0) = 0$, $y(2) = 4$. The extremal corresponding to these boundary conditions is the straight line $y = 2x$, which is included in the central field of extremals $y_1 =$

$= C_1 x$ (C_1 is an arbitrary constant) with centre at the point O (0, 0).

Example 6. Consider the elementary variational problem

$$J[y] = \int_{-1}^{1} y' \cdot \left(2x - \frac{1}{2} y'\right) dx,$$

$$y(-1) = 0, \quad y(1) = \frac{1}{2}$$

The solution of the Euler equation is of the form $y = x^2 + C_1 x + C_2$. The extremal of this problem, $y = x^2 +$

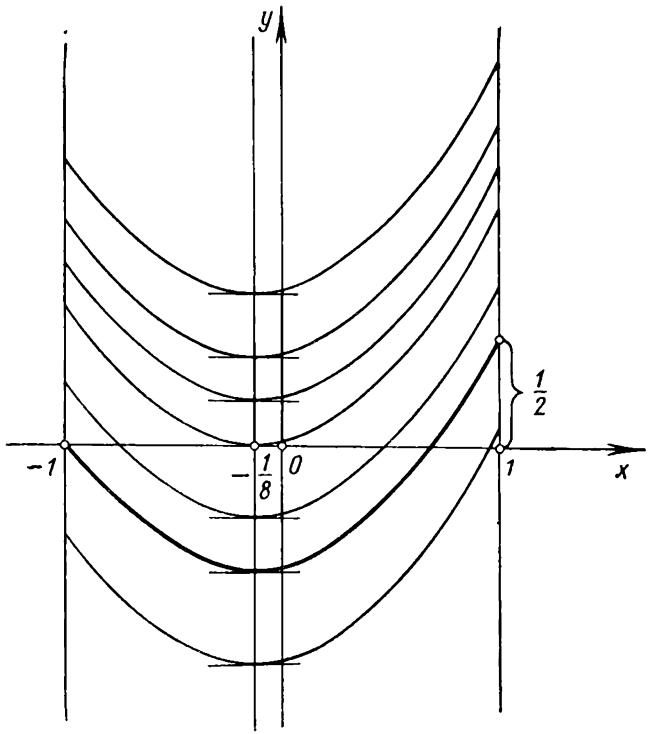

Fig. 9

$+ \frac{x}{4} - \frac{3}{4}$, may be included in the proper field of extremals $y = x^2 + \frac{x}{4} + C_2$ (Fig. 9).

Show that the extremals of the following elementary variational problems may be included in the extremal field (proper or central).

126. $J[y] = \int_0^1 (y'^2 - 2xy)\,dx;\ y(0) = y(1) = 0.$

127. $J[y] = \int_0^1 (2e^x y + y'^2)\,dx;\ y(0) = 1, \quad y(1) = e.$

128. $J[y] = \int_0^a (y^2 - y'^2)\,dx\ (a > 0,\ a \neq k\pi);$

$$y(0) = 0, \quad y(a) = 0.$$

129. $J[y] = \int_0^2 (y'^2 + x^2)\,dx;\ y(0) = 1, \quad y(2) = 3.$

Definition. Suppose we have a family $\Phi(x, y, C) = 0$ of plane curves. The *C-discriminant of this family* is a locus of points defined by the system of equations

$$\left.\begin{aligned} &\Phi(x, y, C) = 0, \\ &\frac{\partial \Phi(x, y, C)}{\partial C} = 0 \end{aligned}\right\} \tag{1}$$

In the general case, the C-discriminant includes the envelopes of the family, the locus of nodal points, and the locus of cusps.

The *envelope of the family* $\Phi(x, y, C) = 0$ is a curve, each point of which touches a certain curve of the given family and each portion of which is touched by an infinitude of curves of the family.

If we have a pencil of curves with centre at the point $A(x_0, y_0)$, then the centre of the pencil belongs to the C-discriminant.

Example 7. Find the C-discriminant of the family of curves $y = (x - C)^2$.

Solution. In this case, equations (1) have the form

$$\left.\begin{aligned} y - (x - C)^2 &= 0, \\ 2(x - C) &= 0 \end{aligned}\right\}$$

whence $y = 0$. It is easy to verify that the curve $y = 0$ is the envelope of the given family. Indeed, at any point $x = x_0$ the curve $y = 0$ has a common tangent with the cor-

responding curve of the family $y = (x - x_0)^2$. Furthermore, no matter how small a portion of the curve $y = 0$ we take, an infinitude of curves of the given family are tangent to it. In the given instance, the C-discriminant consists of the envelope alone.

Find the C-discriminants of the given families in the following examples.

130. $y = Cx + C^2$.

131. $y(C - x) - C^2 = 0$.

132. $(x - C)^2 + y^2 = 1$.

If an arc AB of the curve $y = y(x)$ has a point A^* (different from A) in common with the C-discriminant of

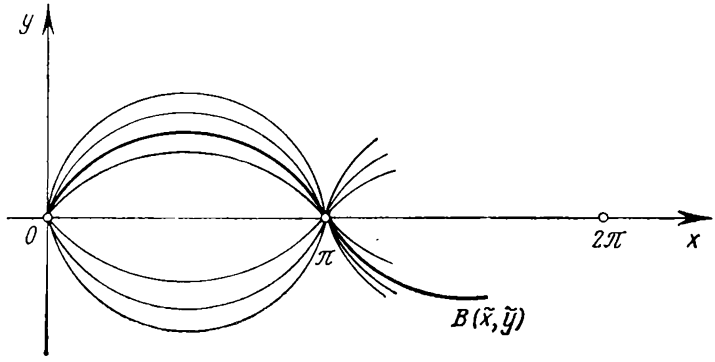

Fig. 10

the pencil $y = y(x, C)$ with centre at the point A, which pencil contains the given curve, then the point A^* is said to be *conjugate* to the point A.

Example 8. Let us consider a one-parameter family of curves $y = C \sin x$. The C-discriminant of this family is defined by the equations

$$\left.\begin{aligned} y - C\sin x &= 0, \\ -\sin x &= 0 \end{aligned}\right\}$$

That is, it is a discrete set of points $(k\pi, 0)$, $k = 0, \pm 1, \pm 2$ (intersection points of the sine curve with the x-axis). For instance, taking $C = 2$, we get the curve $y = 2 \sin x$, which belongs to the given pencil of sine curves centred at the point $O(0, 0)$. If the other end B (Fig. 10) of the arc

of the curve $y = 2 \sin x$ has the abscissa $\tilde{x} \in (\pi, 2\pi)$, then the arc OB will contain another point [in addition to the point O (0, 0)] belonging to the C-discriminant, namely the point O^* $(\pi, 0)$, which will be conjugate to the point O (0, 0). If $0 < x < \pi$, then there will be no points conjugate to the point O (0, 0) on the arc OB.

133. Given a family of curves $y = C \cdot (x - 1)\, x$. Find the point conjugate to the point O (0, 0).

134. Given the family of curves $y = C \cdot \sinh x$. Find the point conjugate to the point O (0, 0).

1°. **Jacobi's sufficient condition for including an extremal in a central field of extremals.** In order to be able to include an arc AB of an extremal in a central field of extremals with centre at the point A (x_0, y_0), it is sufficient that the point A^*, conjugate to the point A, not lie on the arc AB.

Example 9. Let us consider the functional

$$J[y(x)] = \int_0^a (y'^2 - 9y^2 + e^{x^2} - 1)\, dx,$$

$$y(0) = 0, \quad y(a) = 0$$

Verify the possibility of including the extremal $y = 0$ in a central field of extremals with centre at the point $O(0, 0)$.

Solution. The Euler equation of the given functional is of the form $y'' + 9y = 0$. Its general solution is $y(x) = C_1 \sin 3x + C_2 \cos 3x$.

If $a \neq \frac{k\pi}{3}$, k an integer, then the extremal that satisfies the given boundary conditions is the straight line $y = 0$. If we consider the one-parameter family of extremals $y_1 = C_1 \sin 3x$, then, as it is easy to verify, the C-discriminant of this family consists of the points $\left(\frac{k\pi}{3}, 0\right)$, k an integer. Therefore, if $a < \frac{\pi}{3}$, then the extremal $y = 0$ will not have a point conjugate to the point O (0, 0), and then obviously we can include this extremal in the central field of extremals centred at the point O (0, 0). But if $a \geqslant \frac{\pi}{3}$,

then on the extremal $y = 0$ there will be at least one point conjugate to the point O (0, 0), and Jacobi's sufficient condition is not fulfilled. In that case, the extremals $y = C_1 \sin 3x$ do not form a field.

Analytic form of the Jacobi condition. Suppose we have an elementary variational problem:

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\,dx,$$

$$y(x_0) = y_0, \quad y(x_1) = y_1$$

If the solution $u = u(x)$ of the Jacobi equation

$$\left(F_{yy} - \frac{d}{dx} F_{yy'}\right) u - \frac{d}{dx}(F_{y'y'} u') = 0 \qquad (2)$$

that satisfies the condition $u(x_0) = 0$ vanishes at some other point of the interval $x_0 < x < x_1$, then the point A^* that is conjugate to the point $A(x_0, y_0)$ lies on the arc AB of the extremal [the point B has coordinates (x_1, y_1)].

If there is a solution $u(x)$ of the Jacobi equation that satisfies the condition $u(x_0) = 0$ and does not vanish at any point of the half-open interval $x_0 < x \leqslant x_1$, then there are no points conjugate to A on the arc AB. In that case we can include the arc AB of the extremal in the central field of extremals with centre at the point $A(x_0, y_0)$.

It is necessary, in place of $y(x)$, to put the right member of the equation of the extremal $y = y(x, C_0)$ into the functions $F_{yy}(x, y, y')$, $F_{yy'}(y, x, y')$ and $F_{y'y'}(x, y, y')$ in equation (2).

Example 10. Is the Jacobi condition fulfilled for the extremal of the functional

$$J[y(x)] = \int_0^a (y'^2 + x^2)\,dx$$

which extremal passes through the points O (0, 0) and B (a, 3)?

Solution. In the given case, the Jacobi equation is of the form $u'' = 0$. Its general solution is $u(x) = C_1 x + C_2$. From the condition $u(0) = 0$ we find that $C_2 = 0$ so that $u = C_1 x$. These solutions $u = C_1 x$ ($C_1 \neq 0$) do not vanish

for any value $a > 0$. Hence there is no point conjugate to the point O (0, 0) on the arc OB of the extremal. Therefore, it may be included in the central field of extremals with centre at the point O (0, 0). It is easy to verify that the desired extremal is the straight line $y = \frac{3}{a}x$, which clearly can be included in the central field of extremals $y = C_1 x$.

Example 11. Is the Jacobi condition fulfilled for the extremal, passing through the points A (0, 0) and B $(a, 0)$, of the functional

$$J[y] = \int_0^a (y'^2 - 4y^2 + e^{-x^2})\,dx \quad \left(a \neq \left(n + \frac{1}{2}\right)\pi\right)?$$

Solution. The Jacobi equation has the form $u'' + 4u = 0$. Its general solution is

$$u(x) = C_1 \sin 2x + C_2 \cos 2x$$

From the condition $u(0) = 0$ we find that $C_2 = 0$ so that $u(x) = C_1 \sin 2x$. If $a < \frac{\pi}{2}$, then the function $u(x)$ does not vanish for $0 < x \leqslant a$, and the Jacobi condition is fulfilled. But if $a > \frac{\pi}{2}$, then the solution of the Jacobi equation $u = C_1 \sin 2x$ vanishes at the point $x = \frac{\pi}{2}$, which lies in the interval $[0, a]$, and on the arc of the extremal $y = 0$ $(0 \leqslant x \leqslant a)$ there is a point conjugate to the point A (0, 0). Thus, for $a > \frac{\pi}{2}$ there does not exist a central field of extremals that includes the given extremal.

In the following examples, show where the Jacobi condition is fulfilled.

135. $J[y] = \int_{-1}^{1} (12xy + y'^2 + x^2)\,dx;$

$$y(-1) = -2, \quad y(1) = 0.$$

136. $J[y] = \int_0^a (y'^2 + 9y^2 - 3x)\,dx; \quad y(0) = 0, \quad y(a) = 0.$

137. $J[y]=\int_0^1 (1+y'^2)\,dx;\ y(0)=y(1)=0.$

138. $J[y]=\int_0^a y'e^{y'}\,dx;\ y(0)=1,\ y(a)=b\ (a>0).$

139. $J[y]=\int_0^{2\pi} (y'^2-y^2)\,dx;\ y(0)=0,\ y(2\pi)=1.$

140. Show that if the integrand of the functional

$$J[y]=\int_a^b F(x, y')\,dx$$

does not contain y explicitly, then each extremal can always be included in the field of extremals.

Note. The Jacobi condition is necessary for attaining an extremum of the functional $J[y(x)]$; that is, on the extremal AB that realizes an extremum, the point conjugate to A cannot lie in the interval $x_0 < x < x_1$. For example, for the functional

$$J[y]=\int_0^a (y'^4+1)\,dx,\quad y(0)=y(a)=0$$

a minimum is attained on the extremal $y(x)\equiv 0$. On this extremal there are no points conjugate to the point $O(0, 0)$.

Example 12. For the functional

$$J[y]=\int_0^{\frac{5}{4}\pi} (y^2-y'^2)\,dx,\quad y(0)=0,\quad y\left(\frac{5}{4}\pi\right)=0$$

an extremum is not attained on the extremal $y(x)\equiv 0$ because in the interval $\left(0, \frac{5}{4}\pi\right)$ there is a point $O^*(\pi, 0)$ conjugate to the point $O(0, 0)$ $\left($this is because the solution of the Jacobi equation $u''+u=0$, which solution vanishes for $x=0$, is $u(x)=C_1\sin x$, and $u(x)$ also vanishes at the point $x=\pi\in\left(0, \frac{5}{4}\pi\right)\right)$.

Indeed, for a curve "close" to $y \equiv 0$ let us take the curve $y_n(x) = \frac{\sin \frac{4}{5} nx}{n^2}$, for which the conditions $y(0) = y\left(\frac{5\pi}{4}\right) = 0$ are obviously fulfilled, and $y_n'(x) = \frac{4}{5n} \cos n \frac{4}{5} x$. We then get $J[0] = 0$, and

$$J\left[\frac{\sin \frac{4}{5} nx}{n^2}\right] = \int_0^{\frac{5}{4}\pi} \frac{\sin^2\left(\frac{4n}{5} x\right)}{n^4} dx - \int_0^{\frac{5}{4}\pi} \left(\frac{4}{5n}\right)^2 \cos^2\left(\frac{4n}{5} x\right) dx = \frac{5\pi}{8n^2}\left(\frac{1}{n^2} - \frac{16}{25}\right) < 0$$

for any integral $n \geqslant 2$. Hence, the extremal $y(x) \equiv 0$ does not give the given functional a minimum since there are curves close to $y(x) \equiv 0$ on which the values of the functional are negative. Now let us take the family of curves $y_n(x) = \frac{1}{n} \sin \frac{4}{5} x$ that have proximity of any order with respect to the curve $y(x) \equiv 0$. It is readily seen that

$$J[y_n] = \int_0^{\frac{5}{4}\pi} \frac{\sin^2 \frac{4}{5} x}{n^2} dx - \int_0^{\frac{5}{4}\pi} \frac{16}{25n^2} \cos^2 \frac{4}{5} x \, dx = \frac{9\pi}{40n^2} > 0$$

Hence, the extremal $y(x) \equiv 0$ does not give the functional a maximum either.

141. In the functional

$$J[y] = \int_a^b F(x, y, y') dx$$

let us suppose the integrand F has bounded partial derivatives of third order with respect to the variables y, y' in any bounded range of y and y'. Show that if $y = y(x)$ and

$y = y(x) + \eta(x)$ are two close extremals, then the function $\eta(x)$ satisfies the Jacobi equation to within a quantity of higher order compared with the first-order distance between these extremals:

$$F_{yy}\eta + F_{yy'}\eta' - \frac{d}{dx}(F_{yy'}\eta + F_{y'y'}\eta') = 0$$

2°. Sufficient conditions of Legendre. A sufficient condition for including an extremal of the functional

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx,$$
$$y(x_0) = y_0, \quad y(x_1) = y_1$$

in the field of extremals is the fulfilment of *Legendre's strong condition.*

It consists in demanding the fulfilment of the inequality

$$F_{y'y'} > 0$$

at all points of the extremal under consideration (that is, for all $x \in [x_0, x_1]$).

Example 13. Given the functional

$$J[y] = \int_0^2 (y'^4 + y'^2)\,dx;$$
$$y(0) = 1, \quad y(2) = 5$$

The extremals are the straight lines $y = C_1x + C_2$. The desired extremal satisfying the given boundary conditions is the straight line $y = 2x + 1$.

In the given case, $F_{y'y'} = 12y'^2 + 2$, and at all points of the extremal $y = 2x + 1$ we have $F_{y'y'} = 50 > 0$. Legendre's strong condition is fulfilled and, hence, the extremal $y = 2x + 1$ may be included in the extremal field.

This is also evident directly. The extremal $y = 2x + 1$ lies in the one-parameter family of extremals $y = 2x + \alpha$ (α a parameter) that form a proper field.

Example 14. Given the functional

$$\int_{-1}^{1} (x^2y'^2 + 12y^2)\,dx,$$

$$y(-1) = -1, \quad y(1) = 1$$

Solution. The Euler equation for this functional is of the form

$$x^2y'' + 2xy' - 12y = 0$$

Its general solution is

$$y = C_1x^3 + C_2x^{-4}$$

The extremal $y = x^3$ satisfies the given boundary conditions. It cannot be included in the field. The only one-parameter family of extremals that contains it is the family $y = \alpha x^3$. This field does not cover the region containing the point with abscissa $x = 0$ (the extremals of this family do not pass through points of the y-axis with ordinates different from zero).

In the given case, $F_{y'y'} = 2x^2$ and the Legendre condition is not fulfilled for $x = 0$.

Check the possibility of including an extremal in the field for the following functionals.

142. $J[y] = \int_0^1 (y'^2 - yy'^3)\,dx$; $y(0) = 0$, $y(1) = 0$.

143. $J[y] = \int_0^a y'^3\,dx$; $y(0) = 0$, $y(a) = b > 0$.

144. $J[y] = \int_{x_0}^{x_1} n(y)\sqrt{1 + y'^2(x)}\,dx$;

$$y(x_0) = y_0, \quad y(x_1) = y_1, \quad n(y) > 0.$$

145. $J[y] = \int_0^a (6y'^2 - y'^4)\,dx$;

$$y(0) = 0, \quad y(a) = b, \quad a > 0, \quad b > 0.$$

8. SUFFICIENT CONDITIONS FOR THE EXTREMUM OF A FUNCTIONAL

We consider the most elementary variational problem for the functional

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \tag{1}$$

$$y(x_0) = y_0, \quad y(x_1) = y_1 \tag{2}$$

1°. Sufficient conditions of Weierstrass. The *Weierstrass function* $E(x, y, p, y')$ is a function defined by the equation

$$E(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p) F_p(x, y, p) \tag{3}$$

where $p = p(x, y)$ is the slope of the extremal field of the variational problem (1)-(2) under consideration at the point (x, y).

Sufficient conditions for a weak extremum.

A curve C gives the functional (1) *a weak extremum if*:

1. *The curve C is an extremal of the functional* (1) *which satisfies the boundary conditions* (2); *that is, for the functional* (1) *it is a solution of Euler's equation that satisfies the conditions* (2).

2. *The extremal C may be included in the field of extremals*; *this will occur in particular if the Jacobi condition is fulfilled.*

3. *The Weierstrass function $E(x, y, p, y')$ must preserve sign at all points (x, y) close to the extremal C and for values of y' close to $p(x, y)$. The functional $J[y]$ will have a maximum on C if $E \leqslant 0$ and a minimum if $E \geqslant 0$.*

Sufficient conditions for a strong extremum.

A curve C gives the functional (1) *a strong extremum if*:

1. *The curve C is an extremal of the functional* (1) *that satisfies the boundary conditions* (2).

2. *The extremal C may be included in the field of extremals.*

3. *The Weierstrass function $E(x, y, p, y')$ preserves sign at all points (x, y) close to the extremal C and for arbitrary values of y'. There will be a maximum for $E \leqslant 0$ and a minimum for $E \geqslant 0$.*

Note. The Weierstrass condition is necessary for the existence of an extremum in the following meaning: if for

certain values of y' at the points of the extremal the function E has opposite signs, then a strong extremum is not attained. If this property occurs for values of y' arbitrarily close to p, then a weak extremum is not attained either.

Example 1. Test for an extremum the functional

$$J[y] = \int_0^1 (y'^3 + y')\,dx, \quad y(0) = 0, \quad y(1) = 2$$

Solution. The Euler equation for the given functional is of the form $y'y'' = 0$ so that the extremals are the straight lines $y(x) = C_1x + C_2$. The straight line $y = 2x$ is the extremal that satisfies the given boundary conditions. The slope of the field at the points of this extremal is $p = 2$. It is obvious that the given extremal $y = 2x$ is included in the central field of extremals $y = Cx$ with centre at the point O (0, 0). It is also easy to verify that in the given case the Jacobi condition is fulfilled. In the given case, the Jacobi equation has the form $\frac{d}{dx}(6y'u') = 0$, where we have $y' = 2$ by virtue of the equation of the extremal. Hence, the Jacobi equation is of the form $u''(x) = 0$, whence $u(x) = C_1x + C_2$. From the condition $u(0) = 0$ we get $C_2 = 0$. Since for $C_1 \neq 0$, except the point $x = 0$, this solution $u = C_1x$ does not vanish anywhere, the Jacobi condition is fulfilled.

We form the Weierstrass function:

$$E(x, y, p, y') = y'^3 + y' - p^3 - p - (y'-p)(3p^2+1)$$
$$= (y' - p)^2(y' + 2p)$$

The first factor is always nonnegative for arbitrary y', while the second one is positive for values of y' close to 2. Hence, all the conditions for the existence of a weak minimum are fulfilled. However, as is readily evident, if $y' < -4$, the function E will be negative, and the sufficient condition for a strong extremum is not fulfilled, since under the conditions of a strong extremum it is required that the Weierstrass function E preserve sign for all values of y'. Taking into account the note on page 102, we conclude that in this case there is no strong extremum.

Example 2. Investigate for an extremum the functional

$$J[y] = \int_0^1 \left(x + 2y + \frac{1}{2}y'^2\right) dx,$$

$$y(0) = 0, \quad y(1) = 0$$

Solution. The Euler equation for this functional is of the form $y'' = 2$. The extremals are parabolas $y = x^2 + C_1x + C_2$. The extremal satisfying the boundary conditions is $y = x^2 - x$. We set up the Jacobi equation $\frac{d}{dx}(u') = 0$ or $u'' = 0$. Its general solution is $u(x) = C_1x + C_2$. The condition $u(0) = 0$ yields $C_2 = 0$, and since for $C_1 \neq 0$ $u(x) = C_1x$ does not vanish anywhere on the interval $[0, 1]$ except at the point $x = 0$, the Jacobi condition is fulfilled and, hence, the extremal $y = x^2 - x$ may be included in the central field of extremals with centre at the point $O(0, 0)$, namely: $y = x^2 + Cx$ (Fig. 11). The Weierstrass function is of the form $E(x, y, p, y') = \frac{1}{2}(y' - p)^2$, whence it is seen that for arbitrary values of y' we will have $E = \frac{1}{2}(y' - p)^2 \geqslant 0$. Hence, on the extremal $y = x^2 - x$ the given functional attains a strong minimum equal to $J[x^2 - x] = \frac{1}{3}$.

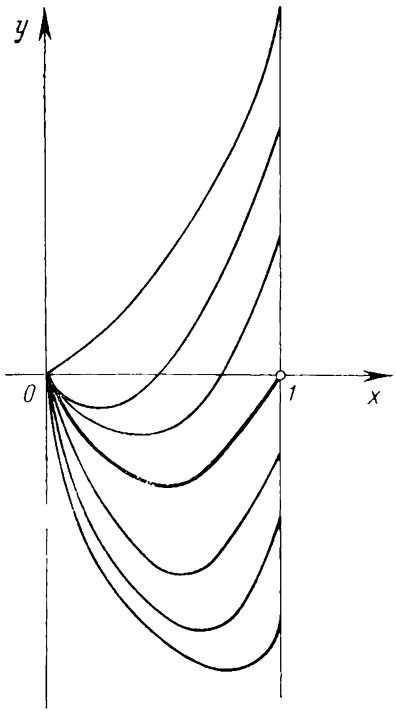

Fig. 11

Test for extrema the following functionals.

146. $J[y] = \int_0^1 e^x \left(y^2 + \frac{1}{2}y'^2\right) dx$; $y(0) = 1$, $y(1) = e$.

147. $J[y]=\int_0^1 e^y y'^2\,dx;\ y(0)=0,\ y(1)=\ln 4.$

148. $J[y]=\int_1^2 \frac{x^3}{y'^2}\,dx;\ y(1)=1,\ y(2)=4.$

149. $J[y]=\int_0^a \frac{dx}{y'};\ y(0)=0,\ y(a)=b,\ a>0,\ b>0.$

150. $J[y]=\int_0^1 (1+x)\,y'^2\,dx;\ y(0)=0,\ y(1)=1.$

151. $J[y]=\int_0^{\pi/2} (y^2-y'^2)\,dx;\ y(0)=1,\ y\left(\frac{\pi}{2}\right)=1.$

152. $J[y]=\int_{-1}^2 y'\,(1+x^2y')\,dx;\ y(-1)=1,\ y(2)=4.$

153. $J[y]=\int_{-1}^1 (y'^3+y'^2)\,dx;\ y(-1)=-1,\ y(1)=3.$

2°. Sufficient conditions of Legendre. Let a function $F(x, y, y')$ have a continuous partial derivative $F_{y'y'}(x, y, y')$ and let the extremal C be included in the field of extremals.

If on the extremal C we have $F_{y'y'}>0$, then a weak minimum is attained on the curve C; if $F_{y'y'}<0$ on the extremal C, then a weak maximum of the functional (1) is attained on it. These conditions are known as *Legendre's strong conditions.*

When $F_{y'y'}(x, y, y')\geqslant 0$ at points (x, y) close to the extremal C, for arbitrary values of y', we have a strong minimum; when $F_{y'y'}(x, y, y')\leqslant 0$ for the indicated values of the arguments, we have a strong maximum.

Example 3. Test for an extremum the functional

$$J[y]=\int_0^1 (y'^3-\alpha y')\,dx,\ y(0)=0,\ y(1)=-2$$

(α any real number).

Solution. Since the integrand depends only on y', the extremals are the straight lines $y = C_1x + C_2$. The extremal satisfying the boundary conditions is the straight line $y = -2x$, which may be included in the central field of extremals $y = Cx$. On this extremal, the slope of the field is $p = -2$. We then find $F_{y'y'} = 6y'$. On the given extremal we have $F_{y'y'} = -12 < 0$, that is, a weak maximum of the functional is attained on the curve $y = -2x$. The sign of $F_{y'y'}$ is not preserved in the case of arbitrary values of y'; hence, the sufficient conditions of a strong maximum are not fulfilled.

In the given case, the Weierstrass function $E(x, y, p, y')$ is of the form

$$E(x, y, p, y') = (y' - p)^2 (y' + 2p)$$

and for certain values of y' it has opposite signs. Taking into account the note on page 102, we find that there is no strong maximum.

Example 4. Test for an extremum the functional

$$J[y] = \int_0^2 (e^{y'} + 3)\, dx, \quad y(0) = 0, \quad y(2) = 1$$

Solution. The extremals are the straight lines $y = C_1x + C_2$. The extremal that satisfies the boundary conditions is the straight line $y = \frac{x}{2}$. It can be included in the central field of extremals $y = Cx$. In the given case, $F_{y'y'}(x, y, y') = e^{y'} > 0$ for any values of y'. Hence, the functional has a strong minimum on the extremal $y = \frac{x}{2}$.

Example 5. Test for an extremum the functional

$$J[y] = \int_0^a \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\, dx, \quad y(0) = 0, \quad y(a) = y_1$$

Solution. The integrand does not depend explicitly on x; hence, we get $F - y' \cdot Fy' = \widetilde{C}_1$ or, in our case,

$$\frac{\sqrt{1 + y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1 + y'^2}} = \widetilde{C}_1$$

whence

$$\frac{1}{\sqrt{y}\sqrt{1+y'^2}}=\widetilde{C}_1 \quad \text{or} \quad y(1+y'^2)=C_1$$

where $C_1=\frac{1}{\widetilde{C}_1^2}$. Set $y'=\cot\frac{t}{2}$. We then have

$$y=C_1\sin^2\frac{t}{2}=\frac{C_1}{2}(1-\cos t)$$

Furthermore,

$$dx=\frac{dy}{\cot\frac{t}{2}}=\frac{C_1\sin t\,dt}{2\cot\frac{t}{2}}=C_1\sin^2\frac{t}{2}\,dt$$

Integrating, we get

$$x=C_1\int\frac{(1-\cos t)\,dt}{2}=\frac{C_1}{2}(t-\sin t)+C_2$$

Thus,

$$\left.\begin{aligned} x&=\widetilde{C}_1(t-\sin t)+C_2,\\ y&=\widetilde{C}_1(1-\cos t)\end{aligned}\right\}$$

are the parametric equations of a family of cycloids. From the condition $y(0)=0$ we find that $C_2=0$. The pencil of cycloids

$$\left.\begin{aligned} x&=C(t-\sin t),\\ y&=C(1-\cos t)\end{aligned}\right\}$$

forms a central field with centre at the point $O(0, 0)$, which field includes the extremal

$$\left.\begin{aligned} x&=R(t-\sin t),\\ y&=R(1-\cos t)\end{aligned}\right\}$$

where R is determined from the condition of passage of a cycloid through the second boundary point $B(a, y_1)$ if $a<2\pi R$ (Fig. 12).

Let us use the Legendre condition. We have

$$F_{y'y'}\quad\frac{1}{\sqrt{y}(1+y'^2)^{3/2}}>0$$

for any values of y'. Hence, for $a < 2\pi R$ on the cycloid

$$\left.\begin{aligned} x &= R(t - \sin t), \\ y &= R(1 - \cos t) \end{aligned}\right\}$$

the given functional has a strong minimum.

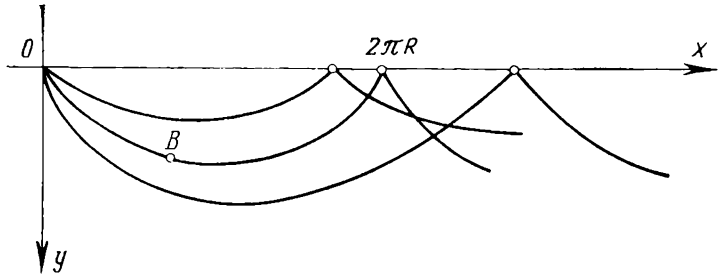

Fig. 12

Utilizing the Legendre condition, test the following functionals for extrema.

154. $J[y] = \int_0^1 (y'^2 + x^2)\,dx;\ y(0) = -1,\ y(1) = 1.$

155. $J[y] = \int_2^3 \frac{x^3}{y'^2}\,dx;\ y(2) = 4,\ y(3) = 9.$

156. $J[y] = \int_1^2 (xy'^4 - 2yy'^3)\,dx;\ y(1) = 0,\ y(2) = 1.$

157. $J[y] = \int_0^a (1 - e^{-y'^2})\,dx;\ y(0) = 0,\ y(a) = b\ (a > 0).$

158. $J[y] = \int_0^1 yy'^2\,dx;\ y(0) = p > 0,\ y(1) = q > 0.$

159. Test for an extremum the functional

$$J[y] = \int_0^1 (\varepsilon y'^2 + y^2 + x^2)\,dx; \quad y(0) = 0, \quad y(1) = 1$$

for various values of the parameter ε.

Example 6. (Euler's problem.) A rod of length l rests on its ends and is subject to a pressure P. The rod is deflected longitudinally at a given value of P (Euler's critical force). It is required to determine the smallest force P that yields a longitudinal deflection.

Solution. Let E be the modulus of elasticity, I is the smallest moment of inertia of the cross sections of the rod, ρ is the radius of curvature, and φ is the angle of the tangent with the axis.

The potential energy of deflection is given by the formula

$$U_1 = \frac{1}{2} EI \int_0^l \frac{dS}{\rho^2}$$

When an end of the rod dips by the quantity

$$\sigma = \int_0^l (1 - \cos\varphi)\, dS$$

the potential energy of the rod decreases by

$$U_2 = P\sigma = Pl - P \int_0^l \cos\varphi\, dS$$

If the potential energy prior to the deformation was zero, then after the deformation it is expressed by the formula

$$U = U_1 - U_2 = \int_0^l \left(\frac{1}{2} EI \cdot \frac{1}{\rho^2} + P\cos\varphi \right) dS - Pl$$

Since $\rho = \frac{dS}{d\varphi}$ and (in the case of small values of φ) $\cos\varphi \approx$ $\approx 1 - \frac{\varphi^2}{2}$, it follows that

$$U = \frac{1}{2} \int_0^l \left[EI \left(\frac{d\varphi}{dS} \right)^2 - P\varphi^2 \right] dS \approx \frac{1}{2} \int_0^l \left[EI \left(\frac{d\varphi}{dx} \right)^2 - P\varphi^2 \right] dx$$

In the case of equilibrium, the potential energy assumes a minimal value. Therefore, the solution of the problem

reduces to determining the minimum of the integral

$$J[\varphi] = \int_0^l \left[EI\left(\frac{d\varphi}{dx}\right)^2 - P\varphi^2\right] dx$$

In the given case,

$$F = EI\left(\frac{d\varphi}{dx}\right)^2 - P\varphi^2$$

and Euler's equation assumes the form

$$\varphi'' + \alpha^2\varphi = 0, \text{ where } \alpha^2 = \frac{P}{EI}$$

The complete integral of this equation is

$$\varphi = C_1 \sin \alpha x + C_2 \cos \alpha x$$

Since we have $\tan \varphi \approx \varphi$ for small values of φ and, besides, $\tan \varphi = y'$, it follows that

$$y' = C_1 \sin \alpha x + C_2 \cos \alpha x$$

whence

$$y(x) = -\frac{C_1 \cos \alpha x}{\alpha} + \frac{C_2 \sin \alpha x}{\alpha} + C$$

If the lower end of the rod lies at the coordinate origin, then for $x = 0$ we will have $y = 0$ and, hence, $C_1 = C = 0$ and

$$y(x) = \frac{C_2}{\alpha} \sin \alpha x$$

Let us see whether the Legendre and Jacobi conditions are fulfilled. The Legendre condition is clearly fulfilled:

$$\frac{\partial^2 F}{\partial \varphi'^2} = 2EI > 0$$

Jacobi's equation is

$$EIz'' + Pz = 0 \text{ or } z'' + \alpha^2 z = 0$$

and $z(0) = 0$. Therefore, the solution of Jacobi's equation is

$$z = A \sin \alpha x$$

The function z vanishes for $x_k = \frac{k\pi}{\alpha} (k = 1, 2, \ldots)$ so that the Jacobi condition will be fulfilled if $l \geqslant \frac{\pi}{\alpha}$, whence

$$P \geqslant \frac{\pi^2}{l^2} EI$$

The smallest value of Euler's critical force will be

$$P_{\min} = \frac{\pi^2}{l^2} EI$$

Here,

$$y = \frac{C_2}{\alpha} \sin \frac{\pi x}{l}$$

is the equation of the deflection curve.

3°. **Figuratrix.** Suppose we have a functional

$$J[y] = \int_a^b F(x, y, y')\, dx$$

We will consider x and y to be parameters and we will examine the function $Y = F(x, y, y')$ as a function of the argument y'. The graph of this function in the plane of the variables (y', Y) is called a *figuratrix*. It is easy to show that the Weierstrass function $E(x, y, p, y')$ is the difference between the figuratrix and the tangent to it drawn at the point with abscissa $y' = p$. The fact that the Weierstrass function preserves sign for certain values of y' means that the figuratrix lies above the tangent or under it for the indicated values of y'. In that case we have a weak extremum. If the figuratrix lies to one side of the tangent for all values of y' and for values of the parameters x and y close to the points of the extremal, then we have a strong extremum.

In these terms, the sufficient Legendre condition reads as follows: *if for all points (x, y) close to the extremal the figuratrix is everywhere convex or concave, then we have a strong extremum.*

Example 7. Test for an extremum the functional

$$J[y] = \int_0^a y'^2\, dx \,(a > 0), \quad y(0) = 0, \quad y(a) = b, \quad b > 0$$

Solution. The extremals are the straight lines $y = C_1x + C_2$. The desired extremal is defined by the equation $y = \frac{b}{a}x$. It is included in the central field of extremals. In the given case, the figuratrix is a parabola, $Y = y'^2$ (Fig. 13). It is easy to see that the figuratrix lies entirely

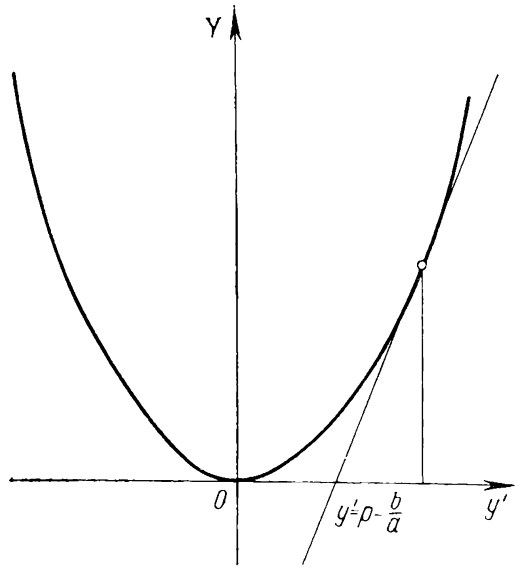

Fig. 13

above the tangent drawn to it at the point $p = \frac{b}{a}$ for any a and b, $a \neq 0$. Hence, the extremal $y = \frac{b}{a}x$ gives the functional a strong minimum.

Example 8. Test for an extremum the functional

$$J[y] = \int_0^a y'^3\,dx, \quad y(0) = 0, \quad y(a) = b, \quad b > 0$$

Solution. The sought-for extremal is the straight line $y = \frac{b}{a}x$, which is included in the central field of extremals $y = Cx$ with centre at the point O (0, 0). The figuratrix is a cubic parabola, $Y = y'^3$ (Fig. 14). For values of y'

sufficiently close to the value $p = \frac{b}{a}$, the figuratrix lies above the tangent to it drawn at the point with abscissa $y' = \frac{b}{a}$. From Fig. 14 it may be seen that the figuratrix intersects the tangent at the point with abscissa $y' = -\frac{2b}{a}$

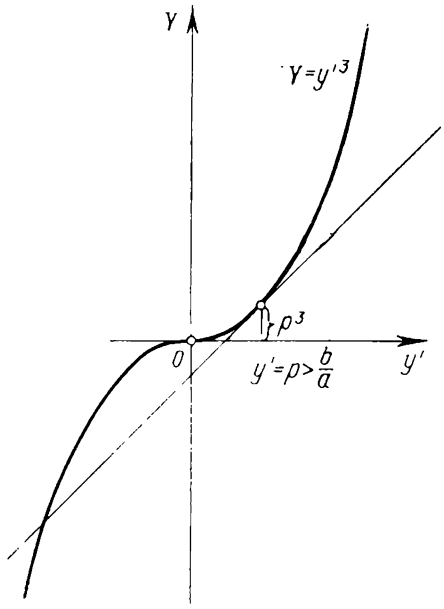

Fig. 14

and lies above the tangent to the left of this point. Hence, a weak minimum is attained on the extremal $y = \frac{b}{a} x$.

Note that if $p = 0$ (this corresponds to the case $b = 0$, the extremal is a portion of the x-axis), then the tangent to the figuratrix is the y'-axis, and the point O (0, 0) itself is an inflection point of the figuratrix. Taking into account the note on page 102, we see that in an arbitrarily small neighbourhood of the point $O(0, 0)$ the figuratrix has positive and negative ordinates. Hence, the Weierstrass function E has opposite signs for values of y' arbitrarily close to $p = 0$ and, consequently, in this case a weak extremum is not attained either.

Example 9. Show that the extremal $y = 0$ of the variational problem

$$J[y] = \int_0^1 (y'^2 - yy'^3)\,dx, \quad y(0) = y(1) = 0$$

gives the functional a weak minimum.

Solution. The Legendre condition in this case yields

$$F_{y'y'}|_{y=0} = (2 - 6yy')|_{y=0} = 2 > 0$$

In other words, a weak minimum is attained on the extremal $y = 0$. We will show that a strong minimum is not attained

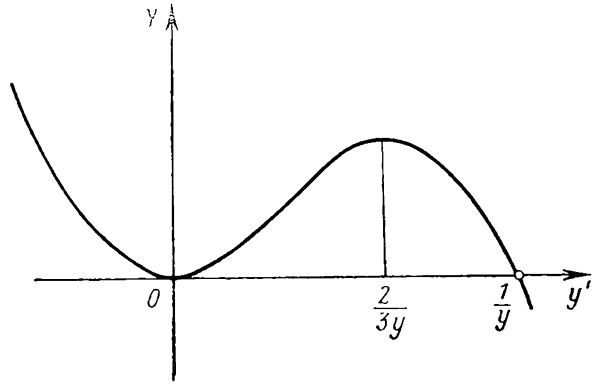

Fig. 15

on it. Let us construct the figuratrix $Y = y'^2 - yy'^3$ for values of $y > 0$ (Fig. 15). From the figure it is clear that the tangent to the figuratrix drawn at the point with abscissa $p = 0$ intersects the figuratrix at the point $y' = \frac{1}{y}$. Thus, for the points (x, y), where $y > 0$, that are close to the points of the extremal $y = 0$, the Weierstrass function E is positive for values of y' less than $\frac{1}{y}$ and is negative for $y' > \frac{1}{y}$. According to the note on page 102, there is no strong minimum. A similar situation occurs for $y < 0$ as well.

This example is noteworthy in that fulfilment of the condition $F_{y'y'} > 0$ on the extremal for arbitrary y' does not imply the existence of a strong extremum.

Using a figuratrix, test the following functionals for extrema.

160. $J[y]=\int_0^1 (1+x)\,y'^2\,dx;\ y(0)=0,\ y(1)=-2.$

161. $J[y]=\int_{-1}^{2} y'\,(1+x^2y')\,dx;\ y(-1)=y(2)=1.$

162. $J[y]=\int_0^a (1-e^{-y'^4})\,dx;$

$$y(0)=0,\ y(a)=b \quad (a>0,\ b>0).$$

163. $J[y]=\int_0^a (6y'^2-y'^4+yy')\,dx;$

$$y(0)=0,\ y(a)=b \quad (a>0,\ b>0).$$

Note. A sufficient condition for an extremum with respect to the second variation.

Nonnegativity of the second variation is necessary but not sufficient for a functional $J[y]$ to attain a minimum on a given curve.

Example 10. Let us consider the functional

$$J[y]=\int_0^1 y^2(x)\,[x-y(x)]\,dx$$

in the space $C(0, 1)$. Euler's equation is of the form $F_y=0$ or $y=0$. The second variation of the functional on the extremal $y=0,\ 0\leqslant x\leqslant 1$,

$$\delta^2 J[0,\ \delta y]=\int_0^1 x\,(\delta y)^2\,dx$$

is positive for every $\delta y\not\equiv 0$. However, the functional $J[y]$ also assumes negative values in any neighbourhood of zero; for a given $\varepsilon>0$ it suffices to take the function

$$y_\varepsilon(x)=\begin{cases} -x+\varepsilon, & 0\leqslant x<\varepsilon,\\ 0, & x\geqslant\varepsilon \end{cases}$$

Then $J[y_\varepsilon]=-\frac{\varepsilon^4}{6}<0$ for any $\varepsilon>0$.

Definition. A quadratic functional $L_2(h)$ specified on some normed space is said to be *strong-positive* if there exists a constant $k > 0$ such that

$$L_2(h) \geqslant k \| h \|^2$$

for all h.

A sufficient condition for a minimum. *For a functional $J[y]$ defined in a normed space to have a minimum at a stationary point $y = y_0$, it is sufficient that when $y = y_0$ the second variation be strong-positive*, that is, that the following condition be fulfilled:

$$\delta^2 J[y_0, \delta y] \geqslant k \| \delta y \|^2$$

where k = constant, $k > 0$.

4°. Let us seek the extremum of the functional

$$J[y_1, y_2, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx \qquad (4)$$

dependent on n functions $y_1(x)$, $y_2(x)$, . . ., $y_n(x)$ with the boundary conditions

$$y_k(x_0) = y_{k0}, \quad y_k(x_1) = y_{k1} \quad (k = 1, 2, \ldots, n) \qquad (5)$$

Legendre's strong condition consists in the requirement that the following inequalities hold true

$$F_{y'y'} > 0,$$

$$\begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} \end{vmatrix} > 0, \ \ldots, \ \begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} & \ldots & F_{y_1'y_n'} \\ F_{y'y_1'} & F_{y_2'y_2'} & \ldots & F_{y_2'y_n'} \\ \ldots & \ldots & \ldots & \ldots \\ F_{y_n'y_1'} & F_{y_n'y_2'} & \ldots & F_{y_n'y_n'} \end{vmatrix} > 0 \qquad (6)$$

at all points of the indicated extremal of the functional (4).

Jacobi's strong condition consists in the requirement that the interval $[x_0, x_1]$ not contain a point conjugate to the point x_0.

Legendre's strong condition (6) in conjunction with Jacobi's strong condition ensure the existence of at least a weak minimum of the functional (4).

Example 11. Test for an extremum the functional

$$J[y, z] = \int_0^1 (y'^2 + z'^2)\, dx, \tag{7}$$

$$\left.\begin{aligned} y(0) &= 0, \quad z(0) = 0, \\ y(1) &= 1, \quad z(1) = 2 \end{aligned}\right\} \tag{8}$$

Solution. Euler's equations for the functional (7) are

$$y'' = 0, \quad z'' = 0$$

so that

$$\left.\begin{aligned} y(x) &= C_1 + C_2 x, \\ z(x) &= C_3 + C_4 x \end{aligned}\right\}$$

Utilizing the conditions (8), we get

$$C_1 = 0, \quad C_2 = 1, \quad C_3 = 0, \quad C_4 = 2$$

The desired extremal

$$\left.\begin{aligned} y(x) &= x, \\ z(x) &= 2x \end{aligned}\right\} \tag{9}$$

is a straight line passing through the origin.

We have

$$F_{y'y'} = 2, \quad F_{y'z'} = 0, \quad F_{z'y'} = 0, \quad F_{z'z'} = 2$$

Legendre's strong condition is fulfilled:

$$F_{y'y'} = 2 > 0, \quad \begin{vmatrix} F_{y'y'} & F_{y'z'} \\ F_{z'y'} & F_{z'z'} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \tag{10}$$

Let us now check to see whether Jacobi's strong condition is fulfilled or not.

One of the definitions of a conjugate point is the following (see [7]). Suppose we have a family of extremals of the functional (4) issuing from an initial point $(x_0, y_{10}, \ldots, y_{n0})$ in close-lying but linearly independent directions.

The point $x^* \in [x_0, x_1]$ is said to be *conjugate to the point* x_0 if there exists a sequence of extremals issuing from an initial point and arbitrarily close to the given extremal such that each of these extremals intersects the given extre-

mal, and the abscissas of the intersection points converge to the point x^*.

In the given example, the extremals are the straight lines (9). All the extremals issuing from the point (0, 0, 0) intersect the extremal (9) only at that point. Hence, the interval [0, 1] of variation of x does not contain a point conjugate to the point $x_0 = 0$. Thus, both Legendre's strong condition and Jacobi's strong condition are fulfilled, so that the extremal (9) gives the functional (7) a weak minimum.

Test the following functionals for extrema.

164. $J[y(x), z(x)] = \int_0^1 \sqrt{1 + y'^2 + z'^2}\, dx;$

$$y(0) = 0,\ y(1) = 2,\ z(0) = 0,\ z(1) = 4.$$

165. $J[y(x), z(x)] = \int_0^1 (y'^2 + z'^2 + 4z)\, dx;$

$$y(0) = 0,\ y(1) = 1,\ z(0) = 0,\ z(1) = 0.$$

9. CONDITIONAL EXTREMUM

1°. Isoperimetric problem. Let there be given two functions $F(x, y, y')$ and $G(x, y, y')$.

From among all curves $y = y(x) \in C_1[x_0, x_1]$ along which the functional

$$K[y] = \int_{x_0}^{x_1} G(x, y, y')\, dx$$

assumes a given value l, determine the one for which the functional

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

assumes an extremal value.

With respect to the functions F and G we assume that they have continuous first and second partial derivatives for $x_0 \leqslant x \leqslant x_1$ and for arbitrary values of the variables y, y'.

Euler's theorem. *If a curve* $y = y(x)$ *extremizes the functional*

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx \tag{1}$$

subject to the conditions

$$K[y] = \int_{x_0}^{x_1} G(x, y, y')\, dx = l, \quad y(x_0) = y_0, \quad y(x_1) = y_1 \tag{2}$$

and if $y = y(x)$ *is not an extremal of the functional* K, *then there exists a constant* λ *such that the curve* $y = y(x)$ *is an extremal of the functional*

$$L = \int_{x_0}^{x_1} [F(x, y, y') + \lambda G(x, y, y')]\, dx \tag{3}$$

Example 1. (Dido's problem.) Among closed curves of length $2l$, **find the one that bounds the largest area.**

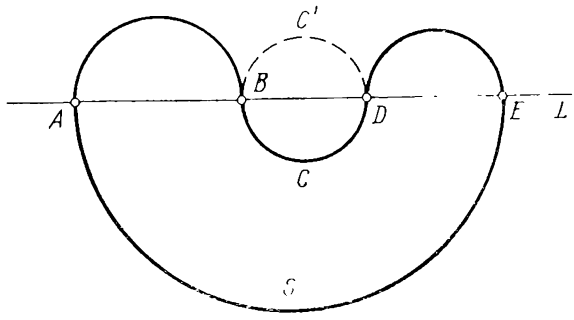

Fig. 16

Solution. First of all, note that the curve must be convex. Indeed, if that were not so, there would be a straight line L (Fig. 16) such that if a portion of the boundary BCD is reflected in the line, then we obtain a region of greater area than the original region having the same length of the boundary line.

Further note that any straight line that bisects a closed curve bounding a maximum area will also divide the area

in half. Suppose the opposite. Suppose a straight line L_1 does not have this property. Then, by making a mirror reflection about L_1 of that portion of the figure with the greater area, we would obtain a curve of the same length but it would bound a greater area.

Choosing for the x-axis any one of the straight lines that bisects the curve, we arrive at the following statement of the problem.

Find a curve $y = y(x)$, $y(-a) = y(a) = 0$, which together with the segment $-a \leqslant x \leqslant a$, for a given length $l > 2a$, bounds a maximum area. Thus, the problem has reduced to seeking the extremum of the functional

$$J[y(x)] = \int_{-a}^{a} y(x)\,dx, \quad y(-a) = y(a) = 0 \tag{4}$$

subject to the accessory condition that

$$K[y(x)] = \int_{-a}^{a} \sqrt{1 + y'^2(x)}\,dx = l \quad (l > 2a) \tag{5}$$

We form the auxiliary function

$$H = F + \lambda G = y(x) + \lambda \sqrt{1 + y'^2(x)} \tag{6}$$

and consider the auxiliary functional

$$L = \int_{-a}^{a} H(x, y, y')\,dx \tag{7}$$

Euler's equation for the functional (7) is

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1$$

whence

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + C_1$$

Solving this equation for y', we get

$$\frac{dy}{dx} = \frac{x + C_1}{\sqrt{\lambda^2 - (x + C_1)^2}} \tag{8}$$

Integrating equation (8), we get

$$(x + C_1)^2 + (y + C_2)^2 = \lambda^2$$

which is a circle of radius λ with centre at the point $(-C_1, -C_2)$. We determine the constants C_1, C_2 and the parameter λ from the boundary conditions $y(-a) = y(a) = 0$ and the isoperimetric condition (5). We have

$$\left.\begin{aligned} C_2^2 &= \lambda^2 - (C_1 - a)^2, \\ C_2^2 &= \lambda^2 - (C_1 + a)^2 \end{aligned}\right\}$$

whence

$$C_1 = 0, \quad C_2 = \sqrt{\lambda^2 - a^2}$$

so that

$$y = \sqrt{\lambda^2 - x^2} - \sqrt{\lambda^2 - a^2} \quad \text{and} \quad y' = -\frac{x}{\sqrt{\lambda^2 - x^2}}$$

Then condition (5) yields

$$l = \int_{-a}^{a} \frac{\lambda\, dx}{\sqrt{\lambda^2 - x^2}} = \lambda \arcsin \frac{x}{\lambda} \Big|_{x=-a}^{x=a} = 2\lambda \arcsin \frac{a}{\lambda}$$

or

$$\frac{a}{\lambda} = \sin \frac{l}{2\lambda}$$

Solving this transcendental equation for λ, we find a certain value $\lambda = \lambda_0$ and then also the quantity $C_2 = \sqrt{\lambda_0^2 - a^2}$.

It is easy to see that the equation $\frac{a}{\lambda} = \sin \frac{l}{2\lambda}$ always has a solution. Indeed, setting $\frac{l}{2\lambda} = t$, we reduce this equation to the form $\sin t = \frac{2a}{l} t$, where $\frac{2a}{l} = \alpha < 1$ by the statement of the problem. At the point $t = 0$, the function $y = \sin t$ has the tangent slope $\frac{\pi}{4}$, while the function $y = \alpha t$ has a smaller slope. Hence, the graphs of these functions have at least one point of intersection other than $O(0, 0)$.

The reciprocity law of isoperimetric problems. The extremals of the functional

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

subject to the accessory condition

$$K[y(x)] = \int_{x_0}^{x_1} G(x, y, y')\, dx = \text{constant}$$

coincide with the extremals of the functional $K[y(x)]$ under the condition $J[y(x)] = \text{constant}$.

Using the reciprocity law, we obtain the following result from Dido's problem: *of all closed curves bounding a given area, the one of shortest length is a circle.*

This result is readily obtainable directly if one takes advantage of the parametric form of the variational problem.

Let

$$\left.\begin{aligned} x &= x(t), \ [x(t_0) = x(t_1)], \\ y &= y(t), \ [y(t_0) = y(t_1)] \end{aligned}\right\} \ (t_0 \leqslant t \leqslant t_1)$$

be the equations of an arbitrary closed curve. The problem reduces to seeking the extremum of the functional

$$\int (\dot{x}^2 + \dot{y}^2)^{1/2}\, dt$$

subject to the condition that

$$\int (x\dot{y} - y\dot{x})\, dx = C$$

Introducing the function

$$F = (\dot{x}^2 + \dot{y}^2)^{1/2} + \lambda (x\dot{y} - y\dot{x})$$

we find (see page 74) that for the curve yielding an extremum the curvature $\frac{1}{r}$ is constant:

$$\frac{1}{r} = \lambda$$

Hence, the desired extremal is a circle.

The law of reciprocity makes it possible to solve certain "variational" problems of elementary geometry without any computations.

Example 2. Show that (1) of all triangles having a given base and a given perimeter, the isosceles triangle has the largest area; (2) the isosceles triangle has the smallest perimeter for a given area and a given base.

Solution. (1) Take an ellipse whose foci are the ends of the bases of the triangles under consideration (Fig. 17). From

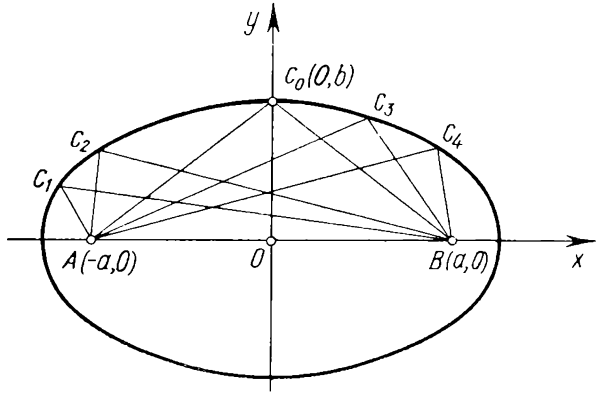

Fig. 17

the property of an ellipse we conclude that all the triangles ACB have the same perimeter. It is clear that the triangle with greatest height will have the largest area, which corresponds to the case when the vertex of the triangle coincides with the vertex C_0 of the ellipse. The triangle AC_0B is then an isosceles triangle.

(2) According to the law of reciprocity, an isosceles triangle has the shortest perimeter for a given area and a given base.

Example 3. Find the minimum of the integral

$$J[y] = \int_0^{\pi} y'^2(x)\,dx$$

subject to the condition

$$\int_0^{\pi} y^2(x)\,dx = 1, \quad y(0) = y(\pi) = 0$$

Solution. Form the auxiliary functional

$$L[y] = \int_0^{\pi} (y'^2 + \lambda y^2)\, dx$$

and write down the Euler equation for it:

$$2\lambda y - \frac{d}{dx}(2y') = 0 \quad \text{or} \quad y'' - \lambda y = 0 \tag{9}$$

The characteristic equation is $r^2 - \lambda = 0$ or $r_{1,2} = \pm \sqrt{\lambda}$. It is clear that λ must be less than zero since if we assume $\lambda > 0$, then the general solution of (9) will have the form $y = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$ and the boundary conditions $y(0) = y(\pi) = 0$ will only be satisfied for $C_1 = 0$, $C_2 = 0$, that is, for $y(x) \equiv 0$. But then the condition $\int_0^{\pi} y^2(x)\, dx = 1$ will not be fulfilled. Similarly in the case of $\lambda = 0$; the function $y(x) \equiv 0$ will be a solution of Euler's equation (9) that satisfies the given boundary conditions. And so we assume $\lambda < 0$ so that $r_{1,2} = \pm i \sqrt{-\lambda}$ and the general solution of equation (9) is $y = C_1 \sin \sqrt{-\lambda}\, x + C_2 \cos \sqrt{-\lambda}\, x$. The condition $y(0) = 0$ yields $C_2 = 0$, and the condition $y(\pi) = 0$ yields $-\lambda = k^2$ ($k = 1, 2, \ldots$). Thus, $y(x) = C_1 \sin kx$, where C_1 has not yet been determined. Taking advantage of the constraint condition $\int_0^{\pi} y^2(x)\, dx = 1$, we get

$$\int_0^{\pi} C_1^2 \sin^2 kx\, dx = 1$$

whence $C_1 = \pm \sqrt{\frac{2}{\pi}}$. Hence, $y(x) = \pm \sqrt{\frac{2}{\pi}} \sin kx$. But of the extremals $y = \pm \sqrt{\frac{2}{\pi}} \sin kx$ that pass through the points $(0, 0)$ and $(\pi, 0)$, only two satisfy the Jacobi condition, namely, $y(x) = \pm \sqrt{\frac{2}{\pi}} \sin x$. On these extre-

mals,

$$J[y] = \int_0^{\pi} y'^2(x)\,dx = \int_0^{\pi} \frac{2}{\pi}\cos^2 x\,dx = 1$$

Example 4. (Kelvin's problem.) Let an xy-plane be covered with a mass having a continuous density $\mu(x, y)$ and let there be given on the plane a piecewise-smooth curve C and on it two points P_1 and P_2.

From among all curves of a given length l connecting the points P_1 and P_2, find the one which together with the arc P_1P_2 of the curve C bounds a domain D having maximum mass. The points P_1 and P_2 may be coincident.

Solution. Introduce the function

$$V(x, y) = \int \mu(x, y)\,dx$$

Then, by Green's formula,

$$\iint_D \mu(x, y)\,dx\,dy = \iint_D \frac{\partial V}{\partial x}\,dx\,dy = \oint_\Gamma V\,dy$$

where the contour Γ consists of the curve L and the portion P_2P_1 of the given curve C. The integral along this latter portion has a known value which we will denote by K. Therefore, assuming that the curve L is given parametrically as

$$\left.\begin{aligned} x &= x(t), \\ y &= y(t) \end{aligned}\right\} \quad (t_0 \leqslant t \leqslant t_1)$$

we get

$$\iint_D \mu(x, y)\,dx\,dy = \int_{t_0}^{t_1} V(x, y)\,\dot{y}\,dt + K$$

Thus, the problem has reduced to finding the maximum of the functional

$$J_L = \int_{t_0}^{t_1} V(x, y)\,\dot{y}\,dt$$

subject to the condition

$$\int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = l$$

Let us introduce an auxiliary function

$$F = V\dot{y} + \lambda \sqrt{\dot{x}^2 + \dot{y}^2}$$

and take advantage of the Weierstrassian form of Euler's equation. We have

$$F_{xy\cdot} = \frac{\partial V}{\partial x}, \quad F_{yx\cdot} = 0, \quad F_1 = \frac{F_{xx\cdot\cdot}}{\dot{y}^2} = \frac{\lambda}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

so that Euler's equation in Weierstrass' form will look like

$$\frac{1}{r} = \frac{1}{\lambda} \frac{\partial V}{\partial x}$$

or, taking into account the expression for the function $V(x, y)$,

$$\frac{1}{r} = \frac{\mu(x, y)}{\lambda}$$

where r is the radius of curvature of the desired curve.

When $\mu(x, y) = \text{constant}$, we find that the curvature of the desired curve is constant and, hence, the extremals are circles. It is clear that they maximize the functional J_L.

The term *isoperimetric problems* is also used to denote variational problems in which it is required to determine the extremum of the functional

$$J[y] = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx \quad (10)$$

given what are called the isoperimetric conditions:

$$\int_{x_0}^{x_1} G_i(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx = l_i \quad (11)$$

$$(i = 1, 2, \ldots, m)$$

where l_i are constants.

In order to obtain the basic necessary condition in an isoperimetric problem for finding the extremum of the functional (10) subject to the constraints (11), it is necessary to set up an auxiliary functional:

$$\Phi[y] = \int_{x_0}^{x_1} \left(F + \sum_{i=1}^{m} \lambda_i G_i\right) dx \tag{12}$$

where λ_i are constants, and to write down Euler's equations for it. The arbitrary constants $C_1, C_2, \ldots, C_{2n}$ in the general solution of a system of Euler's equations and the constants $\lambda_1, \lambda_2, \ldots, \lambda_m$ are determined from the boundary conditions

$$y_k(x_0) = y_{k0}, \quad y_k(x_1) = y_{k1} \quad (k = 1, 2, \ldots, n)$$

and from the isoperimetric conditions (11):

$$\int_{x_0}^{x_1} G_i\, dx = l_i \quad (i = 1, 2, \ldots, m)$$

Example 5. Find an extremal in the isoperimetric problem of the extremum of the functional

$$J[y(x), z(x)] = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z)\, dx,$$

$$y(0) = 0, \quad z(0) = 0, \quad y(1) = 1, \quad z(1) = 1$$

subject to the condition

$$\int_0^1 (y'^2 - xy' - z'^2)\, dx = 2 \tag{13}$$

Solution. We form the auxiliary functional

$$\Phi = \int_0^1 [y'^2 + z'^2 - 4xz' - 4z + \lambda(y'^2 - xy' - z'^2)]\, dx$$

and for it write down the following system of Euler's equations:

$$\frac{d}{dx}(2y' + 2\lambda y' - \lambda x) = 0,$$

$$4 + \frac{d}{dx}(2z' - 4x - 2\lambda z') = 0$$

Solving this system, we obtain

$$y(x) = \frac{\lambda x^2 + 2C_1 x}{4(1+\lambda)} + C_2,$$

$$z(x) = \frac{C_3 x}{2(1-\lambda)} + C_4$$

The boundary conditions yield

$$C_1 = \frac{3\lambda + 4}{2}, \quad C_2 = 0, \quad C_3 = 2(1-\lambda), \quad C_4 = 0$$

so that

$$y(x) = \frac{\lambda x^2 + (3\lambda + 4)x}{4(1+\lambda)},$$

$$z(x) = x$$

To find λ, we take advantage of the isoperimetric condition (13). Since $y'(x) = \frac{2\lambda x + 3\lambda + 4}{4(1+\lambda)}$ and $z'(x) = 1$, we obtain

$$\int_0^1 \left[\frac{(2\lambda x + 3\lambda + 4)^2}{16(1+\lambda)^2} - \frac{(2\lambda x + 3\lambda + 4)x}{4(1+\lambda)} - 1 \right] dx = 2$$

whence, after simple but unwieldy computations we get an equation for determining λ:

$$\frac{1}{3}(23\lambda^2 + 46\lambda + 24) = 48(\lambda^2 + 2\lambda + 1)$$

whence $\lambda_1 = -\frac{10}{11}$ and $\lambda_2 = -\frac{12}{11}$. Substituting into (13), we are convinced that λ_2 does not satisfy the isoperimetric condition, but λ_1 does.

The desired extremal is determined by the equations

$$\left.\begin{aligned} y(x) &= \frac{7x - 5x^2}{2}, \\ z(x) &= x \end{aligned}\right\}$$

Example 6. Let a rod of length l be fixed at the points (x_0, y_0) and (x_1, y_1). From the theory of elasticity we know that the potential energy of a rod in a deformed state is proportional to the integral, taken along the rod, of the square of its curvature. For the independent variable we take the length s of the rod reckoned from the point (x_0, y_0); we denote by $\theta(s)$ the angle between the tangent to the rod and the x-axis. The curvature will be expressed by the derivative $\theta'(s)$, and the integral whose extremum is sought has the form

$$J = \int_0^l [\theta'(s)]^2 \, ds$$

We know that

$$dx = \cos\theta \, ds, \quad dy = \sin\theta \, ds$$

Thus, we have the following equations of constraint:

$$\int_0^l \cos\theta \, ds = x_1 - x_0, \quad \int_0^l \sin\theta \, ds = y_1 - y_0 \tag{14}$$

Besides, the fact that the rod is fixed is equivalent to specifying the function $\theta(s)$ for $s = 0$ and $s = l$:

$$\theta(0) = a, \quad \theta(l) = b \tag{15}$$

Let us form the Lagrangian function

$$\Phi(\theta, \theta') = [\theta'(s)]^2 + \lambda_1 \cos\theta + \lambda_2 \sin\theta$$

The function Φ does not contain the independent variable s and so we can write down at once the first integral of Euler's equation:

$$\theta'^2 = C + \lambda_1 \cos\theta + \lambda_2 \sin\theta \tag{16}$$

Let us introduce the new constants

$$h = C + \sqrt{\lambda_1^2 + \lambda_2^2}, \quad k^2 = \frac{2\sqrt{\lambda_1^2 + \lambda_2^2}}{C + \sqrt{\lambda_1^2 + \lambda_2^2}}$$

and in place of θ the new variable

$$\varphi = \frac{\theta - \theta_0}{2}, \text{ where } \theta_0 = \arctan\frac{\lambda_2}{\lambda_1}$$

Now, (16) reduces to the form

$$\frac{d\varphi}{ds} = \frac{\sqrt{h}}{2} \sqrt{1 - k^2 \sin^2 \varphi}$$

whence we obtain

$$s = \frac{2}{\sqrt{h}} \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} + s_0$$

The constants h, k^2, θ_0 and s_0 must be determined from the conditions (14) and (15).

The Cartesian coordinates of the points of the rod are found as follows:

$$dx = \cos \theta \, ds = \cos (2\varphi + \theta_0) \, ds,$$
$$dy = \sin \theta \, ds = \sin (2\varphi + \theta_0) \, ds$$

or, by virtue of the fact that

$$ds = \frac{2d\varphi}{\sqrt{h} \sqrt{1 - k^2 \sin^2 \varphi}}$$

we get

$$dx = \frac{2 \cos (2\varphi + \theta_0)}{\sqrt{h (1 - k^2 \sin^2 \varphi)}} \, d\varphi,$$

$$dy = \frac{2 \sin (2\varphi + \theta_0)}{\sqrt{h (1 - k^2 \sin^2 \varphi)}} \, d\varphi$$

whence x and y are determined by means of quadratures.

Find the extremals in the following isoperimetric problems.

166. *The problem of the position of equilibrium of a heavy homogeneous thread under the action of gravity.*

Of all plane curves of length l, whose ends lie at specified points $M_0 (x_0, y_0)$ and $M_1 (x_1, y_1)$, find the one whose ordinate of the centre of gravity is a minimum.

167. $J [y (x)] = \int_0^1 y'^2 (x) \, dx; \quad y (0) = 1, \quad y (1) = 6$

subject to the condition $\int_0^1 y (x) \, dx = 3$.

168. $J[y(x)] = \int_0^1 (x^2 + y'^2(x))\,dx; \quad y(0) = 0, \quad y(1) = 0$

subject to the condition $\int_0^1 y^2(x)\,dx = 2$.

169. $J[y(x)] = \int_0^1 y'^2(x)\,dx; \quad y(0) = 0, \quad y(1) = \frac{1}{4}$

subject to the condition $\int_0^1 [y(x) - y'^2(x)]\,dx = \frac{1}{12}$.

2°. The Lagrangian problem of finding the extremum of a functional $J[y_1, \ldots, y_n]$ subject to the condition that certain constraints are imposed on the functions upon which the functional J depends is also a variational problem involving a conditional extremum.

The problem is this: to find the extremum of the functional

$$J = \int_{x_0}^{x_1} F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\,dx, \tag{17}$$

$$y_j(x_0) = y_{j0}, \quad y_j(x_1) = y_{j1} \quad (j = 1, \ldots, n)$$

subject to the conditions

$$\varphi_i(x, y_1, \ldots, y_n) = 0 \quad (i = 1, \ldots, m;\ m < n) \tag{18}$$

which are taken to be independent.

Theorem. *The functions* $y_1, y_2, \ldots, y_n$ *that extremize the functional* (17) *under the conditions* (18) *satisfy the Euler equations formed for the functional*

$$J^* = \int_{x_0}^{x_1} \left[F + \sum_{i=1}^{m} \lambda_i \varphi_i\right] dx$$

in the case of an appropriate choice of factors $\lambda_i(x)$ $(i = 1, 2, \ldots, m)$.

For the sake of brevity we will write $F + \sum_{i=1}^{m} \lambda_i \varphi_i = \Phi(x, y_1, \ldots, y_n)$. Then the functions $\lambda_i(x)$ and $y_i(x)$

are determined from Euler's equations

$$\Phi'_{y_j} - \frac{d}{dx}\Phi'_{y'_j} = 0 \quad (j = 1, \ldots, n)$$

and

$$\varphi_i(x, y_1, \ldots, y_n) = 0 \quad (i = 1, \ldots, m)$$

The equations $\varphi_i = 0$ may be considered to be the Euler equations for the functional J^* if not only the functions $y_1, y_2, \ldots, y_n$ but also the functions $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ are taken to be the arguments of the functional.

Example 7. Find the shortest distance between the points $A(1, -1, 0)$ and $B(2, 1, -1)$ lying on the surface $15x - 7y + z - 22 = 0$.

Solution. We know that the distance between the two points $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$ on the surface $\varphi(x, y, z) = 0$ is found from the formula

$$l = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx$$

where $y = y(x)$, $z = z(x)$.

We have to find the minimum of l provided $\varphi(x, y, z) = 0$. In our case,

$$x_0 = 1, \quad x_1 = 2, \quad \varphi(x, y, z) = 15x - 7y + z - 22$$

We form the auxiliary functional

$$J^* = \int_1^2 \left[\sqrt{1 + y'^2 + z'^2} + \lambda(x)(15x - 7y + z - 22)\right] dx$$

and write down the Euler equations for it:

$$\lambda(x) \cdot (-7) - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0, \tag{19}$$

$$\lambda(x) \cdot 1 - \frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \tag{20}$$

We solve the system of equations (19)-(20), using the constraint condition

$$15x - 7y + z - 22 = 0 \tag{21}$$

The desired functions $y = y(x)$ and $z = z(x)$ satisfy the following boundary conditions:

$$y(1) = -1,\ y(2) = 1,\ z(1) = 0,\ z(2) = -1 \qquad (22)$$

Multiplying equation (20) by 7 and adding it to (19), we get

$$\frac{d}{dx}\left(\frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$$

whence

$$\frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}} = C_1 \qquad (23)$$

From (21) we have

$$z' = 7y' - 15 \qquad (24)$$

Substituting this value of z' into (23) and solving the resulting differential equation, we get $y(x) = \tilde{C}_1 x + C_2$. The boundary conditions (22) yield $\tilde{C}_1 = 2$, $C_2 = -3$ so that

$$y(x) = 2x - 3 \qquad (25)$$

From (24), taking into account (25), we find

$$z(x) = 1 - x \qquad (26)$$

[the boundary conditions are obviously fulfilled for the function (26)]. From (19) or (20) we get $\lambda(x) \equiv 0$. The desired distance is

$$l = \int_1^2 \sqrt{1 + y'^2 + z'^2}\, dx = \sqrt{6}$$

This result stems immediately from geometrical reasoning.

3°. Geodesics. Suppose a surface is given by the vector equation

$$\boldsymbol{r} = \boldsymbol{r}(u, v) \qquad (27)$$

A *geodesic* is a curve of shortest length lying on a given surface and joining two given points of the surface.

The equations of geodesics may be obtained as Euler's equations corresponding to the variational problem of finding the shortest distance on a surface between two given points.

A curve lying on a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$ may be specified by the parametric equations

$$u = u(t), \quad v = v(t) \tag{28}$$

The length of the curve between the points corresponding to the values t_0 and t_1 of the parameter t is

$$J[u, v] = \int_{t_0}^{t_1} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\, dt \tag{29}$$

where E, F, G are coefficients of the first quadratic form of the surface (27); that is,

$$E = \left(\frac{\partial \boldsymbol{r}}{\partial u}, \frac{\partial \boldsymbol{r}}{\partial u}\right), \quad F = \left(\frac{\partial \boldsymbol{r}}{\partial u}, \frac{\partial \boldsymbol{r}}{\partial v}\right), \quad G = \left(\frac{\partial \boldsymbol{r}}{\partial v}, \frac{\partial \boldsymbol{r}}{\partial v}\right) \tag{30}$$

Here, $(\boldsymbol{a}, \boldsymbol{b})$ is the scalar product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

The system of Euler's equations for the functional (29) is of the form

$$\left.\begin{aligned} \frac{E_u u'^2 + 2F_u u'v' + G_u v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt}\frac{2(Eu' + Fv')}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} = 0, \\ \frac{E_v u'^2 + 2F_v u'v' + G_v v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt}\frac{Fu' + Gv'}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} = 0 \end{aligned}\right\} \tag{31}$$

Example 8. From among all curves on a sphere of radius R that join two given points, find the shortest curve (geodesic).

Solution. Let φ be the longitude, θ the latitude of a point on the sphere, and let $\varphi = \varphi(\theta)$ be the equation of the desired curve. We then have

$$\boldsymbol{r} = \boldsymbol{r}(\varphi, \theta) = x(\varphi, \theta)\,\boldsymbol{i} + y(\varphi, \theta)\,\boldsymbol{j} + z(\varphi, \theta)\,\boldsymbol{k}$$

Therefore,

$$E = (\boldsymbol{r}_\varphi, \boldsymbol{r}_\varphi) = R^2 \sin^2\theta,$$

$$G = (\boldsymbol{r}_\theta, \boldsymbol{r}_\theta) = R^2, \quad F = (\boldsymbol{r}_\theta, \boldsymbol{r}_\varphi) = 0$$

whence, by formula (29), we have

$$J[\varphi, \theta] = R\int_{\theta_0}^{\theta_1} \sqrt{d\theta^2 + \sin^2\theta\, d\varphi^2} = R\int_{\theta_0}^{\theta_1} \sqrt{1 + \sin^2\theta \cdot \varphi'^2(\theta)}\, d\theta$$

The integrand does not contain the desired function $\varphi(\theta)$ and so Euler's equation will be

$$\frac{d}{d\theta} f_{\varphi'} = 0, \text{ where } f_{\varphi'} = \frac{\sin^2\theta \cdot \varphi'(\theta)}{\sqrt{1+\sin^2\theta \cdot \varphi'^2(\theta)}}$$

so that

$$\frac{\sin^2\theta \cdot \varphi'(\theta)}{\sqrt{1+\sin^2\theta \cdot \varphi'^2(\theta)}} = C_1$$

From this we get

$$\varphi'(\theta) = \frac{C_1}{\sin\theta \sqrt{\sin^2\theta - C_1^2}} = \frac{C_1}{\sin^2\theta \sqrt{1-\frac{C_1^2}{\sin^2\theta}}}$$

$$= \frac{C_1}{\sin^2\theta \sqrt{(1-C_1^2)-C_1^2\cot^2\theta}} = -\frac{C_1 d(\cot\theta)}{\sqrt{(1-C_1^2)-C_1^2\cot^2\theta}}$$

Integrating we obtain

$$\varphi(\theta) = \arccos \frac{C_1 \cot\theta}{\sqrt{1-C_1^2}} + C_2$$

or

$$\varphi(\theta) = \arccos(C \cdot \cot\theta) + C_2, \text{ where } C = \frac{C_1}{\sqrt{1-C_1^2}}$$

whence

$$C \cdot \cot\theta = \cos[\varphi(\theta) - C_2]$$

or

$$\cot\theta = A\cos\varphi(\theta) + B\sin\varphi(\theta) \tag{32}$$

where

$$A = \frac{\cos C_2}{C}, \quad B = \frac{\sin C_2}{C}$$

Multiplying both members of (32) by $R\sin\theta$, we obtain

$$R\cos\theta = AR\cos\varphi\sin\theta + BR\sin\varphi\sin\theta$$

or, passing to Cartesian coordinates,

$$z = Ax + By$$

This is the equation of the plane passing through the centre of the sphere and intersecting the sphere along a great circle. Thus, the shortest curve (geodesic) is an arc of a great circle.

Example 9. Show that at every point of any geodesic on a surface of revolution the product of the radius of a parallel by the sine of the angle between the geodesic and the meridian is a constant (Clairaut's theorem).

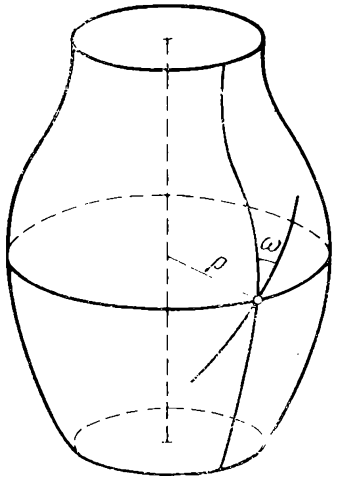

Fig. 18

Solution. The equation of a surface of revolution in cylindrical coordinates has the form

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi,$$
$$z = f(\rho)$$

We find the coefficients E, F and G:

$$E = 1 + f'^2, \quad F = 0, \quad G = \rho^2$$

so that the differential of arc length dS on the surface of revolution is

$$dS = \sqrt{\rho^2 + (1 + f'^2_\rho)\, \rho'^2}\, d\varphi$$

The geodesics on the surface of revolution are extremals of the functional

$$\int_{\varphi_0}^{\varphi_1} \sqrt{\rho^2 + (1 + f'^2_\rho)\, \rho'^2}\, d\varphi$$

The integrand does not contain φ explicitly and so we immediately have

$$\frac{\rho^2}{\sqrt{\rho^2 + (1 + f'^2_\rho)\, \rho'^2}} = \text{constant}$$

or $\rho^2 \frac{d\varphi}{dS} = \text{constant}$. Noting that $\rho \frac{d\varphi}{dS} = \sin \omega$ (Fig. 18), we get $\rho \sin \omega = \text{constant}$, which is what we set out to prove.

170. Find the shortest distance between the points A (1, 0, —1) and B (0, —1, 1) lying on the surface $x + y + z = 0$.

171. Find the geodesics of the circular cylinder $r = R$.

10. MOVING BOUNDARY PROBLEMS

1°. An elementary problem with moving boundaries. Let $F = F(x, y, y')$ be a three times differentiable function of its arguments and in the xy-plane let there be two curves

$$y = \varphi(x) \text{ and } y = \psi(x) \tag{1}$$

where $\varphi(x) \in C_1[a, b]$ and $\psi(x) \in C_1[a, b]$.

We consider the functional

$$J[y] = \int_{\gamma} F(x, y, y')\, dx \tag{2}$$

defined on the smooth curves $y = y(x)$, the endpoints of which $A(x_0, y_0)$ and $B(x_1, y_1)$ lie on the given curves (1) so that $y_0 = \varphi(x_0)$, $y_1 = \psi(x_1)$. It is required to find the extremum of the functional (2).

Theorem. *Let the curve* γ: $y = y(x)$ *extremize the functiona*

$$J[\gamma] = \int_{\gamma} F(x, y, y')\, dx$$

from among all curves of the class C_1 *joining two arbitrary points of two given curves* $y = \varphi(x)$, $y = \psi(x)$. *Then the curve* γ *is an extremal and the transversality conditions*

$$\left.\begin{aligned} [F + (\varphi' - y')\, F_{y'}]\,|_{x=x_0} &= 0, \\ [F + (\psi' - y')\, F_{y'}]\,|_{x=x_1} &= 0 \end{aligned}\right\} \tag{3}$$

are fulfilled at the endpoints $A(x_0, y_0)$ *and* $B(x_1, y_1)$ *of the curve* γ.

Thus, to solve an elementary problem with moving boundaries it is necessary:

(1) To write down and solve the appropriate Euler equations. We then obtain a family of extremals $y = f(x, C_1, C_2)$ that is dependent on two parameters C_1 and C_2.

(2) From the transversality conditions (3) and from the equations

$$\left.\begin{aligned} f(x_0, C_1, C_2) &= \varphi(x_0), \\ f(x_1, C_1, C_2) &= \psi(x_1) \end{aligned}\right\} \tag{4}$$

to determine the constants C_1 C_2, X_0, x_1.

(3) To compute the extremum of the functional (2).

Example 1. Find the transversality condition for the functional

$$J[y]=\int_{x_0}^{x_1} f(x,\ y)\, e^{\arctan y'} \sqrt{1+y'^2}\, dx, \tag{5}$$

$$f(x,\ y) \neq 0$$

Solution. Let the left end of the extremal be fixed at the

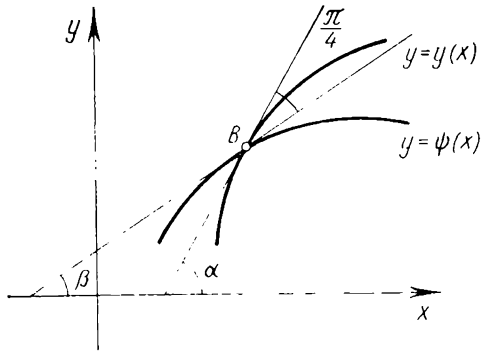

Fig. 19

point $A\ (x_0,\ y_0)$ and let the right end $B\ (x_1,\ y_1)$ be movable along the curve $y = \psi\ (x)$. We then get

$$[F+(\psi'-y')\,F_{y'}]\,|_{x=x_1}=0$$

In our case

$$F=f(x,\ y)\, e^{\arctan y'} \sqrt{1+y'^2},$$

$$F_{y'}=f(x,\ y)\, e^{\arctan y'} \frac{1+y'}{\sqrt{1+y'^2}}$$

The transversality condition is written thus:

$$\left[f(x,\ y)\, e^{\arctan y'} \sqrt{1+y'^2} + (\psi'-y')\, f(x,\ y)\, e^{\arctan y'} \frac{1+y'}{\sqrt{1+y'^2}} \right]\Big|_{x=x_1} = 0$$

From this, by virtue of the condition $f(x, y) \neq 0$, we get

$$\frac{\psi' - y'}{1 + \psi' y'} = -1 \tag{6}$$

Geometrically, condition (6) signifies that the extremals $y = y(x)$ must intersect the curve $y = \psi(x)$, along which the boundary point $B(x_1, y_1)$ slides, at an angle $\frac{\pi}{4}$.

Actually, the relation (6) may be represented thus: suppose that the tangent to the extremal at the point $B(x_1, y_1)$ lying on the curve $y = \psi(x)$ cuts the x-axis at an angle α, while the tangent to the given curve $y = \psi(x)$ cuts it at an angle β (Fig. 19). Then $\tan \alpha = y'$, $\tan \beta = \psi'$ and the left member of (6) yields $\tan(\beta - \alpha)$; but $-1 = \tan\left(-\frac{\pi}{4}\right)$; therefore $\beta - \alpha = -\frac{\pi}{4}$, whence $\alpha = \beta + \frac{\pi}{4}$, which is what we set out to prove.

Example 2. Find the distance between the parabola $y = x^2$ and the straight line $x - y = 5$.

Solution. The problem reduces to finding the extremal value of the integral

$$J = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx \tag{7}$$

subject to the condition that the left end of the extremal can move along the curve $y = x^2$, while the right end can move along the straight line $y = x - 5$. Thus, in our case we have $\varphi(x) = x^2$, $\psi(x) = x - 5$. The general solution of Euler's equation will be $y = C_1 x + C_2$, where C_1 and C_2 are arbitrary constants that have to be determined.

The transversality condition (3) is of the form

$$\left[\sqrt{1 + y'^2} + (2x - y')\frac{y'}{\sqrt{1 + y'^2}}\right]\Bigg|_{x=x_0} = 0,$$

$$\left[\sqrt{1 + y'^2} + (1 - y')\frac{y'}{\sqrt{1 + y'^2}}\right]\Bigg|_{x=x_1} = 0$$

where $y' = C_1$. In our case, equations (4) take the form

$$\left.\begin{aligned} C_1 x_0 + C_2 &= x_0^2, \\ C_1 x_1 + C_2 &= x_1 - 5 \end{aligned}\right\}$$

We thus have a system of four equations in four unknowns C_1, C_2, x_0, x_1:

$$\left.\begin{aligned} \sqrt{1+C_1^2}+(2x_0-C_1)\frac{C_1}{\sqrt{1+C_1^2}}=0,\\ \sqrt{1+C_1^2}+(1-C_1)\frac{C_1}{\sqrt{1+C_1^2}}=0,\\ C_1x_0+C_2=x_0^2,\\ C_1x_1+C_2=x_1-5 \end{aligned}\right\} \tag{8}$$

Solving this system, we get

$$C_1=-1,\ C_2=\frac{3}{4},\ x_0=\frac{1}{2},\ x_1=\frac{23}{8}$$

Hence, the equation of the extremal is $y=-x+\frac{3}{4}$ and the distance between the given parabola and straight line is

$$l=\int_{\frac{1}{2}}^{\frac{23}{8}}\sqrt{1+(-1)^2}\,dx=\sqrt{2}\,x\Big|_{\frac{1}{2}}^{\frac{23}{8}}=\frac{19\sqrt{2}}{8}$$

172. Find the shortest distance between the point A (1, 0) and the ellipse $4x^2+9y^2=36$.

173. Find the shortest distance from the point A (−1, 5) to the parabola $y^2=x$.

174. Find the shortest distance between the circle $x^2+y^2=1$ and the straight line $x+y=4$.

175. Find the shortest distance between the point A (−1, 3) and the straight line $y=1-3x$.

176. Prove that for a functional of the form

$$J[y]=\int_{x_0}^{x_1}h(x,\ y)\sqrt{1+y'^2}\,dx$$

where $h(x,\ y)\neq 0$ at the boundary points, the transversality conditions are of the form

$$y'(x)=-\frac{1}{\varphi'(x)}\quad\text{and}\quad y'(x)=-\frac{1}{\psi'(x)}$$

That is, the transversality conditions reduce to orthogonality conditions.

2°. The moving boundary problem for functionals of the form

$$J[y, z] = \int_{x_0}^{x_1} F(x, y, z, y', z')\, dx \tag{9}$$

In testing the functional (9) for an extremum we assume that at least one of the boundary points $A(x_0, y_0, z_0)$ or $B(x_1, y_1, z_1)$ is in motion along the given curve.

The extremum of $J[y, z]$ can only be attained on integral curves of the system of Euler equations

$$\left.\begin{aligned} F_y - \frac{d}{dx} F_{y'} = 0, \\ F_z - \frac{d}{dx} F_{z'} = 0 \end{aligned}\right\} \tag{10}$$

Let the point $A(x_0, y_0, z_0)$ be fixed and let the other boundary point $B(x_1, y_1, z_1)$ be in motion along a certain curve specified by the equations

$$\left.\begin{aligned} y = \varphi(x), \\ z = \psi(x) \end{aligned}\right\} \tag{11}$$

Then the transversality condition in this case assumes the form

$$[F + (\varphi' - y') F_{y'} + (\psi' - z') F_{z'}]|_{x=x_1} = 0 \tag{12}$$

In similar fashion we can write down the transversality condition for the left end as well (if it too is in motion along some curve $\left.\begin{aligned} y = \tilde{\varphi}(x) \\ z = \tilde{\psi}(x) \end{aligned}\right\}$):

$$[F + (\tilde{\varphi}' - y') F_{y'} + (\tilde{\psi}' - z') F_{z'}]|_{x=x_0} = 0$$

Example 3. Find the shortest distance from the point $M(x_0, y_0, z_0)$ to the straight line

$$\left.\begin{aligned} y = mx + p, \\ z = nx + q \end{aligned}\right\}$$

Solution. The problem reduces to finding the extremum (minimum) of the integral

$$J[y, z] = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx \tag{13}$$

subject to the condition that the right end of the extremal can move along the straight line

$$\left.\begin{aligned} y &= mx + p, \\ z &= nx + q \end{aligned}\right\} \tag{14}$$

That is, in our case, the functions φ and ψ have, respectively, the forms

$$\varphi(x) = mx + p, \quad \psi(x) = nx + q$$

The general solution of the appropriate system of Euler equations is

$$\left.\begin{aligned} y &= C_1 x + C_2, \\ z &= C_3 x + C_4 \end{aligned}\right\} \tag{15}$$

where C_i $(i = 1, 2, 3, 4)$ have to be determined.

The transversality condition (on the right extremity) looks like this:

$$\left[\sqrt{1 + y'^2 + z'^2} + (m - y')\frac{y'}{\sqrt{1 + y'^2 + z'^2}} + (n - z')\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right]\Bigg|_{x=x_1} = 0$$

whence, by virtue of the fact that $y' = C_1$, $z' = C_3$, we get

$$1 + mC_1 + nC_3 = 0 \tag{16}$$

The relation (16) expresses the condition of perpendicularity of the desired straight line (15) to the given straight line (14).

We take advantage of the fact that the desired line (15) passes through the point $M(x_0, y_0, z_0)$:

$$y_0 = C_1 x_0 + C_2, \quad z_0 = C_3 x_0 + C_4 \tag{17}$$

and also the fact that the right extremity moves along the line (14):

$$\left.\begin{aligned} C_1 x_1 + C_2 &= mx_1 + p, \\ C_3 x_1 + C_4 &= nx_1 + q \end{aligned}\right\} \tag{18}$$

From the five equations of (16), (17) and (18), determine C_1, C_2, C_3, C_4 and x_1 (x_0, y_0, z_0, m, n, p, q are given numbers). To find the integral (13) it suffices to know x_1, C_1 and C_3. We have

$$x_1 = \frac{x_0 + m(y_0 - p) + n(z_0 - q)}{1 + n^2 + m^2},$$

$$C_1 = \frac{mx_0 + mn(z_0 - q) - (1 + n^2)(y_0 - p)}{m(y_0 - p) + n(z_0 - q) - (m^2 + n^2)x_0},$$

$$C_3 = \frac{nx_0 + mn(y_0 - p) - (1 + m^2)(z_0 - q)}{m(y_0 - p) + n(z_0 - q) - (m^2 + n^2)x_0}$$

Substituting these quantities into (13), we get

$$h = \min J[y, z]$$

$$= \sqrt{x_0^2 + (y_0 - p)^2 + (z_0 - q)^2 - \frac{[x_0 + m(y_0 - p) + n(z_0 - q)]^2}{1 + n^2 + m^2}}$$

If the boundary point A (x_0, y_0, z_0) is fixed and the other boundary point B (x_1, y_1, z_1) can move along some surface $z = \varphi(x, y)$, then the transversality conditions will be

$$\left.\begin{aligned} [F - y'F_{y'} + (\varphi'_x - z')F_{z'}]|_{x=x_1} = 0, \\ [F_{y'} + F_{z'}\varphi'_y]|_{x=x_1} = 0 \end{aligned}\right\} \qquad (19)$$

Generally speaking, the conditions (19) together with the equation $z = \varphi(x, y)$ enable one to determine two arbitrary constants in the general solution of the system of Euler equations [the other two constants are determined from the condition that the extremal passes through the fixed point A (x_0, y_0, z_0)].

If the boundary point A (x_0, y_0, z_0) is the moving point, then for $x = x_0$ we get conditions similar to the conditions (19).

Example 4. Find the shortest distance from the point A (1, 1, 1) to the sphere

$$x^2 + y^2 + z^2 = 1 \qquad (20)$$

Solution. The problem reduces to testing for an extremum the functional

$$J[y, z] = \int_{x_1}^{1} \sqrt{1 + y'^2(x) + z'^2(x)}\, dx \tag{21}$$

where the point $B(x_1, y_1, z_1)$ must lie on the sphere (20). The extremals of the functional (21) are the straight lines

$$\left.\begin{aligned} y &= C_1 x + C_2, \\ z &= C_3 x + C_4 \end{aligned}\right\} \tag{22}$$

From the condition that the extremal (22) passes through the point $A(1, 1, 1)$ we get

$$\left.\begin{aligned} C_1 + C_2 &= 1, \\ C_3 + C_4 &= 1 \end{aligned}\right\} \tag{23}$$

With allowance for (22), the transversality conditions (19) are

$$\left.\begin{aligned} &\left[\sqrt{1 + y'^2 + z'^2} - \frac{y'^2}{\sqrt{1 + y'^2 + z'^2}} + \left(\frac{-x}{\sqrt{1 - x^2 - y^2}} - z'\right) \frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right]\Bigg|_{x=x_1} = 0, \\ &\left[\frac{y'}{\sqrt{1 + y'^2 + z'^2}} + \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \cdot \frac{(-y)}{\sqrt{1 - x^2 - y^2}}\right]\Bigg|_{x=x_1} = 0 \end{aligned}\right\}$$

whence, after a few simple manipulations, we get

$$\left.\begin{aligned} z_1 - C_3 x_1 &= 0, \\ C_1 z_1 - C_3 y_1 &= 0 \end{aligned}\right\} \tag{24}$$

where x_1, y_1, z_1 are the coordinates of the desired point B.

From the condition that the extremal (22) passes through the point $B(x_1, y_1, z_1)$ we have

$$\left.\begin{aligned} y_1 &= C_1 x_1 + C_2, \\ z_1 &= C_3 x_1 + C_4 \end{aligned}\right\} \tag{25}$$

From (23), (24) and (25) we find

$$C_1 = 1, \quad C_2 = 0, \quad C_3 = 1, \quad C_4 = 0$$

so that the equation of the extremal is

$$\left.\begin{aligned} y &= x, \\ z &= x \end{aligned}\right\} \tag{26}$$

Since the point $B(x_1, y_1, z_1)$ must lie on the sphere (20), it follows, making allowance for (26), that $x_1^2 + x_1^2 + x_1^2 = 1$, that is, $x_1 = \pm \frac{1}{\sqrt{3}}$.

We thus get two points:

$$B_1\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and } B_2\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

It is easy to see, from geometrical reasoning, that the extremal (26) joining the point A to the point B_1 minimizes the functional (21). This minimum is

$$J_{\min} = \int\limits_{\frac{1}{\sqrt{3}}}^{1} \sqrt{1+1+1}\, dx = \sqrt{3} - 1$$

The extremal (26) joining the point A and the point B_2 yields the maximum

$$J_{\max} = \int\limits_{-\frac{1}{\sqrt{3}}}^{1} \sqrt{3}\, dx = \sqrt{3} + 1$$

Note 1. In deriving the transversality conditions (24), we took $\varphi(x, y) \equiv \sqrt{1 - x^2 - y^2}$. We can readily verify that the conditions (24) are preserved if $\varphi(x, y) \equiv -\sqrt{1 - x^2 - y^2}$.

Note 2. From geometrical reasoning it is evident that the extremal (26) is orthogonal to the sphere $x^2 + y^2 + z^2 = 1$.

Example 5. Consider the same problem of the extremum of the functional (21), but for A take the centre of the sphere $O(0, 0, 0)$.

Solution. The extremals of the functional are the straight lines (22) and the condition that the extremal pass through the point $O(0, 0, 0)$ immediately yields $C_2 = C_4 = 0$.

The transversality conditions remain the same:

$$\left.\begin{aligned} z_1 - C_3 x_1 &= 0, \\ C_1 z_1 - C_3 y_1 &= 0 \end{aligned}\right\} \tag{27}$$

while the conditions on the moving end will be

$$\left.\begin{aligned} y_1 &= C_1 x_1, \\ z_1 &= C_3 x_1 \end{aligned}\right\} \tag{28}$$

Finally,

$$x_1^2 + y_1^2 + z_1^2 = 1 \tag{29}$$

For determining the five quantities C_1, C_3, x_1, y_1, and z_1 we have the five relations of (27), (28), and (29), of which only three are independent:

$$\left.\begin{aligned} y_1 &= C_1 x_1, \\ z_1 &= C_3 x_1, \\ x_1^2 + y_1^2 + z_1^2 &= 1 \end{aligned}\right\} \tag{30}$$

Using the relations (30), we find

$$x_1 = \pm \frac{1}{\sqrt{1 + C_1^2 + C_3^2}}, \quad y_1 = \pm \frac{1}{\sqrt{1 + C_1^2 + C_3^2}},$$

$$z_1 = \pm \frac{1}{\sqrt{1 + C_1^2 + C_3^2}}$$

where C_1 and C_2 are arbitrary constants.

This arbitrariness is clear from geometrical reasoning: the distance from the point O (0, 0, 0) to the sphere (20) is the same in any direction, that is, for any values of C_1 and C_2.

The value of the functional $J[y, z]$ on the extremals

$$\left.\begin{aligned} y &= C_1 x, \\ z &= C_3 x \end{aligned}\right\}$$

is equal to

$$J[y, z] = \int_0^{\frac{1}{\sqrt{1 + C_1^2 + C_3^2}}} \sqrt{1 + C_1^2 + C_3^2}\, dx = 1$$

Example 6. Find the transversality condition for the functional

$$J[y, z] = \int_{x_0}^{x_1} f(x, y, z) \sqrt{1 + y'^2 + z'^2}\, dx \qquad (31)$$

if the point $A(x_0, y_0, z_0)$ is fixed and the point $B(x_1, y_1, z_1)$ lies on the surface $z = \varphi(x, y)$.

Solution. In this case the transversality conditions are

$$\left.\begin{aligned} (1 + \varphi'_x \cdot z')\,|_{x=x_1} &= 0, \\ (y' + \varphi'_y \cdot z')\,|_{x=x_1} &= 0 \end{aligned}\right\}$$

or

$$\frac{1}{\varphi'_x}\Big|_{x=x_1} = \frac{y'}{\varphi'_y}\Big|_{x=x_1} = \frac{z'}{-1}\Big|_{x=x_1}$$

These are the conditions of parallelism of the tangent vector $\tau\{1, y', z'\}$ to the desired extremal at the point $B(x_1, y_1, z_1)$ with the vector $n\{\varphi'_x, \varphi'_y, -1\}$ of the normal to the surface $z = \varphi(x, y)$ at the same point. Thus, for functionals of the form (31) the transversality conditions reduce to the orthogonality conditions.

177. Show that if the transversality condition coincides for all initial data with the orthogonality condition, then the integrand F has the following structure:

$$F = f(x, y, z) \sqrt{1 + y'^2 + z'^2}$$

where $f(x, y, z)$ is an arbitrary differentiable function of x, y, z.

178. Find the shortest distance from the point $M(0, 0, 3)$ to the surface $z = x^2 + y^2$.

179. Find the shortest distance from the point $M(2, 0, 5)$ to the surface $z = x^2 + y^2$.

180. Find the shortest distance between the surfaces

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 4$$

181. Test for an extremum the functional

$$J[y, z] = \int_0^{x_1} (y'^2 + z'^2 + 2yz)\, dx$$

subject to the conditions $y(0) = 0$, $z(0) = 0$ and the point $B(x_1, y_1, z_1)$ moves over the plane $x = x_1$.

3°. **Geodesic distance.** The magnitude of the integral

$$J[y] = \int_A^B F(x, y, y')\, dx \tag{32}$$

taken along the curve γ from the point A to the point B is called the *J-length* of the curve γ. If γ is an extremal, then $J[y]$ is termed the *geodesic distance between the points A and B*, or the *J-distance*, while the extremal is called the *J-line* (straight line).

Example 7. Find the geodesic distance from the point $A(0, 0)$ to the point $B(1, 1)$ if this distance is determined by means of the functional

$$J[y] = \int_A^B y^2 y'^2\, dx$$

Solution. The geodesic distance from the point A to the point B is the value of the given functional on the extremal connecting these points. Euler's equation is

$$2yy'^2 - \frac{d}{dx}(2y^2y') = 0 \text{ or } yy'' + y'^2 = 0$$

It is easy to see that

$$yy'' + y'^2 = \frac{d}{dx}(yy')$$

so that $2yy' = C_1$ and $y^2 = C_1x + C_2$. Using the boundary conditions $y|_{x=0} = 0$, $y|_{x=1} = 1$, we get $C_1 = 1$, $C_2 = 0$. Thus, the extremal joining the points A and B is the parabola

$$y^2 = x$$

Furthermore, $2yy' = 1$, $yy' = \frac{1}{2}$ and, hence, $(yy')^2 = \frac{1}{4}$. The geodesic distance between the points A and B is, by definition, equal to

$$J(A, B) = \int_0^1 \frac{1}{4}\, dx = \frac{1}{4}$$

Suppose a line $\mathscr{L}$ is given: $\varphi(x, y) = 0$.

The *geodesic distance from point* B, exterior to $\mathscr{L}$, to that line is the geodesic distance from B to $A \in \mathscr{L}$ such that the functional (32) is computed along the extremal γ joining B and A. Note that γ cuts $\mathscr{L}$ at the point A transversally.

A *geodesic circle* (*J-circle*) is a line all points of which lie at the same geodesic distance from a given point. A J-ellipse and J-hyperbola are defined similarly.

Example 8. Find the J-circle having its centre at the point $O(0, 0)$ and with radius R if the geodesic distance is determined by means of the functional

$$J[y] = \int_A^B y^2 y'^2 \, dx$$

Solution. The extremals of the functional cut the geodesic circle transversally. For the extremals we have (see preceding example)

$$y^2 = C_1 x, \quad 2yy' = C_1$$

and, consequently,

$$y' = \frac{y}{2x}$$

From the transversality condition

$$y^2 y' (2\varphi' - y') = 0$$

we find that the slope of the tangent to the J-circle is $\varphi' = \frac{y'}{2}$ and, therefore, the differential equation of the J-circle is $y' = \frac{y}{4x}$, whence the equation of the J-circle is $y^4 = Cx$. In order to find C, note that there is a point (C^3, C) on the geodesic circle, and the equation of the geodesic radius (i.e. of the extremal) passing through this point is $y^2 = \frac{x}{C}$. From this we have $yy' = \frac{1}{2C}$ and, hence,

$$R = \int_0^{C^3} (yy')^2 \, dx = \int_0^{C^3} \frac{1}{4C^2} \, dx = \frac{C}{4}$$

Thus, $C = 4R$ and the geodesic circle of radius R centred at the origin has the equation $y^4 = 4Rx$.

Example 9. Find a J-circle of radius R centred at the point O (0, 0) if the geodesic distance is determined by the functional

$$J[y] = \int_A^B \sqrt{1 + y'^2(x)}\, dx$$

Solution. The extremals of the functional are the straight lines $y = C_1x + C_2$. From the condition that the extremals pass through the point O (0, 0) we find $C_2 = 0$ so that $y = C_1x$ and, hence, $y' = \frac{y}{x}$.

Here, the transversality condition coincides with the orthogonality condition and for this reason the slope of the tangent to the J-circle is $\varphi' = -\frac{1}{y'}$. Consequently, the differential equation of the J-circle is $y' = -\frac{x}{y}$, whence the equation of the J-circle is $x^2 + y^2 = C^2$. On this circle lies the point $(C, 0)$. The equation of the geodesic radius passing through this point is $y = 0$, so that $y' = 0$ and

$$R = \int_0^C dx = C$$

Thus, $C = R$ and the equation of the desired geodesic circle of radius R is the ordinary equation of a circle: $x^2 + y^2 = R^2$.

Note. The concepts that we have just introduced permit speaking of a non-Euclidean geometry with a differential of arc

$$ds = F(x, y, y')\, dx$$

If $F = \sqrt{1 + y'^2(x)}$, then, as we have seen, the J-lines (straight lines) turn into ordinary straight lines and our geometry becomes the conventional Euclidean geometry.

For an arbitrary function F that satisfies only ordinary conditions of continuity and differentiability with respect to all three arguments, the geometry thus introduced does not much resemble ordinary geometry: it is not always possible

to draw a J-line through two points, and it may happen that several J-lines pass through two points and, hence, the J-distance between two points is not a single-valued function of the coordinates.

182. Find the geodesic distance from the point A (0, 0) to the point B (1, 2) if this distance is determined with the aid of the functional

$$J[y] = \int (y^2 + y'^2)\, dx$$

183. Find the geodesic distance from the point A (0, 1) to the point B (1, 1) if this distance is determined by the functional

$$J[y] = \int (12xy + y'^2)\, dx$$

184. Find the J-circle of radius $R = 8$ centred at the point O (0, 0) if the geodesic distance is determined by the functional

$$J[y] = \int y'^3\, dx$$

11. DISCONTINUOUS PROBLEMS. ONE-SIDED VARIATIONS

1°. Discontinuous problems. The extremal $y = y(x)$ of the functional

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx \qquad (1)$$

is a twice continuously differentiable function if the derivative $F_{y'y'}(x, y(x), y'(y))$ does not vanish. However, variational problems occur in which the extremum is attained on a curve which is only piecewise smooth.

(a) *Discontinuous problems of the first kind.* We consider a problem of finding the extremum of the functional (1), assuming that the admissible curves satisfy the boundary conditions

$$y(x_0) = y_0,\ y(x_1) = y_1 \qquad (2)$$

and can have a salient point with abscissa c ($x_0 < c < x_1$). This salient point is only possible where $F_{y'y'} = 0$. At the

salient point, the extremal must satisfy the Weierstrass-Erdman conditions

$$\left.\begin{aligned} F_{y'}|_{x=c-0} - F_{y'}|_{x=c+0} &= 0, \\ (F - y' \cdot F_{y'})|_{x=c-0} - (F - y' \cdot F_{y'})|_{x=c+0} &= 0 \end{aligned}\right\} \qquad (3)$$

Together with the continuity conditions of the desired extremal, they permit determining the coordinates of the salient point.

On each of the two intervals $[x_0, c]$ and $[c, x_1]$, the extremal must satisfy the Euler equation, that is, a second-order differential equation. The solution of these two equations yields four arbitrary constants, which, generally speaking, are found from the boundary conditions (2) and the conditions (3) at the salient point.

Example 1. Find the broken-line extremals (if they exist) of the functional

$$J[y] = \int_0^a (y'^2 - y^2)\, dx$$

Solution. We write down the first of conditions (3) that must hold true at the salient point:

$$F_{y'}|_{x=c-0} = F_{y'}|_{x=c+0} \quad (0 < c < a)$$

In this case it has the form

$$y'(c - 0) = y'(c + 0)$$

and signifies that the derivative $y'(x)$ is continuous when $x = c$. Hence, there are no salient points. This is also evident from the fact that in the given case $F_{y'y'} = 2 > 0$ everywhere. Therefore, in the problem at hand the extremum can be attained only on smooth curves.

Example 2. Find the broken-line extremals of the functional

$$J[y] = \int_0^2 (y'^4 - 6y'^2)\, dx, \quad y(0) = 0, \quad y(2) = 0$$

assuming that y' may have one discontinuity point corresponding to the abscissa $x = c$.

Solution. In this case, $F_{y'y'} = 12y'^2 - 12$ may vanish and therefore the extremal may have salient points. Since

the integrand depends solely on y', the extremals are the straight lines

$$y = C_1x + C_2$$

Set

$$y_- = mx + n \quad (0 \leqslant x < c), \quad y_+ = px + q \quad (c \leqslant x \leqslant 2)$$

From the boundary conditions we find $n = 0$, $q = -2p$, so that

$$y_- = mx, \quad y_+ = p\,(x - 2) \tag{4}$$

The continuity condition of the extremal yields

$$mc = p\,(c - 2) \tag{5}$$

Writing down the Weierstrass-Erdman conditions, we have

$$\left.\begin{aligned} F_{y'} &= 4y'^3 - 12y', \\ F - y' \cdot F_{y'} &= -3y'^4 + 6y'^2 \end{aligned}\right\}$$

Since $y'_- = m$, $y'_+ = p$, we obtain

$$\left.\begin{aligned} 4m^3 - 12m &= 4p^3 - 12p, \\ -3m^4 + 6m^2 &= -3p^4 + 6p^2 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} (m - p)\,(m^2 + mp + p^2 - 3) &= 0, \\ (m^2 - p^2)\,(m^2 + p^2 - 2) &= 0 \end{aligned}\right\} \tag{6}$$

The second equation in (6) immediately yields $m = p$ or $m = -p$ or

$$m^2 + p^2 - 2 = 0$$

The solution $m = p$ must be discarded. The extremal then has a continuous derivative and from the condition (5) we find that $m = 0$, that is, the extremal is a segment of the x-axis.

Thus, the solution of the system (6) reduces to solving the following systems of equations:

$$\left.\begin{aligned} m &= -p, \\ m^2 + mp + p^2 &= 3 \end{aligned}\right\} \tag{7}$$

and

$$\left.\begin{aligned} m^2 + p^2 &= 2, \\ m^2 + mp + p^2 &= 3 \end{aligned}\right\} \tag{8}$$

The solution of system (7) is: $m = \sqrt{3}$, $p = -\sqrt{3}$ and $m = -\sqrt{3}$, $p = \sqrt{3}$. The solution of system (8) yields $m = p$ and must be discarded. Thus, $m = -p$ and the continuity condition (5) yields $c = 1$. Consequently, the desired extremals are

$$y = \begin{cases} \sqrt{3}\,x, & 0 \leqslant x < 1, \\ -\sqrt{3}\,(x-2), & 1 \leqslant x \leqslant 2 \end{cases}$$

and

$$y = \begin{cases} -\sqrt{3}\,x, & 0 \leqslant x < 1, \\ \sqrt{3}\,(x-2), & 1 \leqslant x \leqslant 2 \end{cases}$$

185. Find the extremals with corner point for the functional

$$J[y] = \int_0^2 y'^2 (y'-1)^2\,dx; \quad y(0) = 0, \quad y(2) = 1$$

186. Find the solution with one corner point in the problem of the minimum of the functional

$$J[y] = \int_0^4 (y'-1)^2 (y'+1)^2\,dx; \quad y(0) = 0, \quad y(4) = 2$$

187. Do there exist solutions with corner points in the problem of the extremum of the functional

$$J[y] = \int_{x_0}^{x_1} (y'^2 + 2xy - y^2)\,dx; \quad y(x_0) = y_0, \quad y(x_1) = y_1?$$

188. Find the solution with corner point in the problem of the extremum of the functional

$$J[y] = \int_{-1}^{1} y^2 (1 - y'^2)\,dx; \quad y(-1) = 0, \quad y(1) = 1$$

189. Find the solution with corner point in the problem of the minimum of the functional

$$\int_{\mu_1}^{\mu_2} (y'^4 - 2y'^2)\,dx$$

190. In the problem of the extremum of the functional

$$\int_{(0,\,0)}^{(x_1,\,y_1)} \sin y'\,dx$$

find the continuous solution and also the solution with corner point.

Note. The Weierstrass-Erdman conditions (3) admit the following geometrical interpretation.

Construct a figuratrix, that is, a curve $Y = F(x, y, y')$ as a function of y'.

Then the conditions (3) signify that for values of the parameters $x = c$, $y = c_1$ that correspond to a corner point, the figuratrix must have a common tangent at the points with abscissas $y'_- = y'(c - 0)$ and $y'_+ = y'(c + 0)$.

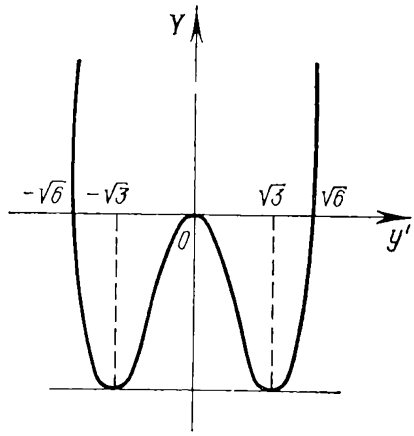

Fig. 20

At the same time we have a pictorial interpretation of the condition $F_{y'y'} \neq 0$ which excludes any possibility of a salient point of the extremals. Indeed, if for example $F_{y'y'} > 0$, then the figuratrix is convex and the tangents to it, drawn at two different points, cannot coincide. And so the extremal in that case cannot have a salient point.

Let us again consider the problem of finding broken-line extremals of a functional (see Example 2 of this section). We have

$$J[y] = \int_0^2 (y'^4 - 6y'^2)\,dx, \quad y(0) = 0, \quad y(2) = 0$$

The extremals are straight lines. The figuratrix $Y = y'^4 - 6y'^2$ here does not depend on the point (x, y). It has a common tangent at points with abscissas $y' = \pm\sqrt{3}$ (Fig. 20). Therefore, the Weierstrass-Erdman conditions will be fulfilled if for the broken-line extremals we

take broken lines whose segments form an angle $\pm \frac{\pi}{3}$ with the x-axis.

On the broken line y_1 with one corner point (Fig. 21) the functional has the value $J[y_1] = -18$. $J[y]$ will have the

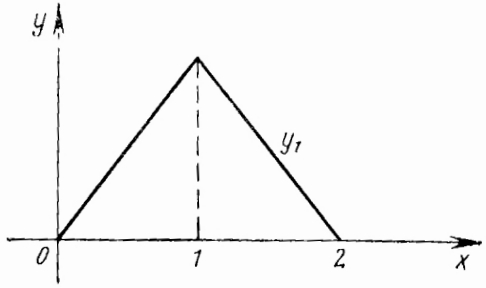

Fig. 21

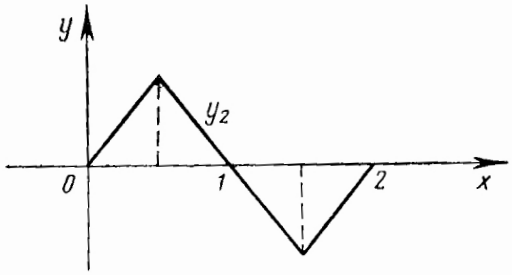

Fig. 22

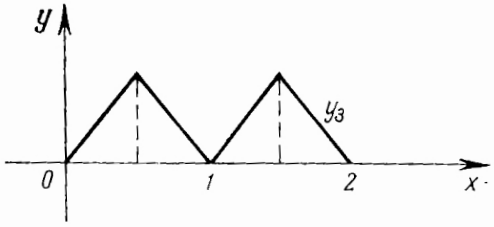

Fig. 23

same value on the broken line y_2 with two corner points (Fig. 22), on the broken line y_3 with three corner points (Fig. 23), and so on.

(b) *Discontinuous problems of the second kind. Discontinuous problems of the second kind* are problems that involve finding

the extremum of a functional

$$J[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx, \tag{9}$$

$$y(x_1) = y_1, \quad y(x_2) = y_2 \tag{10}$$

in which the integrand is discontinuous.

For example, suppose $F(x, y, y')$ suffers a discontinuity along the curve $y = \Phi(x)$ and let $F(x, y, y')$ be equal to $F_1(x, y, y')$ on one side of $y = \Phi(x)$ and be equal to $F_2(x, y, y')$ on the other side.

In the case of a broken-line extremal, this extremal will consist of sections of the extremals $y = y_1(x)$ and $y = y_2(x)$ having the common point $(c, \Phi(c))$ on the line of discontinuity, where $c \in (x_1, x_2)$. To determine the broken-line extremal, we obtain two differential Euler equations, the general solutions of which contain four arbitrary constants C_1, C_2, C_3, C_4. In order to find these constants and also the abscissa c of the meeting point of the extremal and the curve $y = \Phi(x)$ we have: (1) two boundary conditions (10), (2) two conditions that require that the ordinates of the extremities of the extremals at the meeting point be equal to the ordinate of the curve $y = \Phi(x)$ and, finally, (3) the condition at the meeting point:

$$F_1 + (\Phi' - y') F_{1y'} \Big|_{x=c-0} = F_2 + (\Phi' - y') F_{2y'} \Big|_{x=c+0} \tag{11}$$

Generally speaking, these conditions suffice for finding the broken-line extremal.

Example 3. (Problem of the refraction of light.) The velocity of light (constant) in Medium I is v_1, in Medium II it is v_2. Medium I is separated from Medium II by the curve $y = \Phi(x)$.

Derive the law of refraction of a light ray going from point A in Medium I to point B in Medium II if we know that the light ray traverses this path in the shortest time interval.

Solution. The problem reduces to finding the minimum of the integral

$$J = \int_a^c \frac{\sqrt{1 + y'^2}}{v_1}\, dx + \int_c^b \frac{\sqrt{1 + y'^2}}{v_2}\, dx \tag{12}$$

since the first and second integrals in (12) yield the time required for the ray to pass from point A to the line of discontinuity and from that line to point B.

We have a discontinuous problem of the second kind. Here,

$$F_1 = \frac{\sqrt{1+y'^2}}{v_1}, \qquad F_2 = \frac{\sqrt{1+y'^2}}{v_2}$$

Finding the sections of the extremals reduces to finding the extremals of the functional

$$\int \sqrt{1+y'^2}\, dx$$

which, as we know, are straight lines. Hence,

$$y_1 = mx + n, \quad y_2 = px + q$$

Writing down condition (11), we have

$$F_1 - y_1' \frac{\partial F_1}{\partial y_1'} = \frac{\sqrt{1+y_1'^2}}{v_1} - \frac{y_1'^2}{v_1\sqrt{1+y_1'^2}} = \frac{1}{v_1\sqrt{1+y_1'^2}},$$

$$F_2 - y_2' \frac{\partial F_2}{\partial y_2'} = \frac{1}{v_2\sqrt{1+y_2'^2}}$$

Substituting these expressions into (11), we get

$$\frac{1+\Phi' y_1'}{v_1\sqrt{1+y_1'^2}} = \frac{1+\Phi' y_2'}{v_2\sqrt{1+y_2'^2}} \tag{13}$$

Let γ be the angle formed with the x-axis by the tangent to the boundary line at the point c, α, the angle of the left-hand ray with the x-axis, and β, the angle of the right-hand ray. Then $\Phi' = \tan\gamma$, $y_1' = \tan\alpha$, $y_2' = \tan\beta$ and condition (13) assumes the form

$$\frac{1+\tan\alpha\tan\gamma}{v_1\sqrt{1+\tan^2\alpha}} = \frac{1+\tan\beta\tan\gamma}{v_2\sqrt{1+\tan^2\beta}}$$

or

$$\frac{\cos(\gamma-\alpha)}{v_1} = \frac{\cos(\gamma-\beta)}{v_2}$$

where $\gamma - \alpha$ and $\gamma - \beta$ are the angles between the rays and the tangent to the boundary line. Introducing in their place the angles φ and θ between the normal to the boundary

line and the incident and refracted rays, we get

$$\frac{\sin\varphi}{\sin\theta}=\frac{v_1}{v_2}=\text{constant}$$

which is the familiar law of refraction of light.

2°. One-sided variations. We seek the extremum of the functional

$$J[y]=\int_{x_1}^{x_2}F(x,\ y,\ y')\,dx, \tag{14}$$

$$y(x_1)=y_1,\quad y(x_2)=y_2 \tag{15}$$

subject to the condition that

$$y-\varphi(x)\geqslant 0\quad(\text{or } y-\varphi(x)\leqslant 0) \tag{16}$$

(the bounding conditions may be more complicated).

In this case, the desired extremal may consist of sections of the extremals lying in the domain (16) and sections of the boundary $y=\varphi(x)$ of this domain. The desired extremal at meeting points of the indicated sections may be smooth, but it may also have corner points.

The condition at the meeting point is of the form

$$[F(x,\ y,\ y')-F(x,\ y,\ \varphi')-(\varphi'-y')F_{y'}(x,\ y,\ y')]|_{x=\bar{x}}=0$$

If $F_{y'y'}\neq 0$, then at the meeting point $M(\bar{x},\ \bar{y})$ the extremal touches the boundary $y=\varphi(x)$ of the domain.

Example 4. Find the shortest path from point A $(-2,\ 3)$ to point B $(2,\ 3)$ located in the region $y\leqslant x^2$.

Solution. The problem reduces to finding the extremum of the functional

$$J[y]=\int_{-2}^{2}\sqrt{1+y'^2(x)}\,dx \tag{17}$$

subject to the conditions

$$y\leqslant x^2,\quad y(-2)=3,\quad y(2)=3$$

The extremals of the functional (17) are the straight lines

$$y=C_1+C_2x$$

Here,

$$F_{y'y'} = \frac{1}{[1+y'^2(x)]^{3/2}} \neq 0$$

and the desired extremal will consist of portions of the straight lines AM and NB tangent to the parabola $y = x^2$, and of the portion MON of this parabola (Fig. 24). Denote the abscissas of the points of tangency by $\bar{x}$ and $-\bar{x}$ (we take advantage of the symmetry of the problem). At the point of tangency, the ordinates coincide with the slopes of the straight line and the tangent to the parabola, and so we have

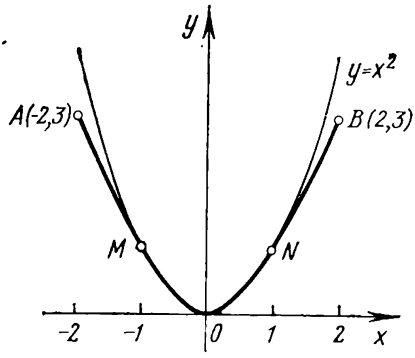

Fig. 24

$$\left.\begin{aligned} C_1 + C_2\bar{x} &= \bar{x}^2, \\ C_2 &= 2\bar{x} \end{aligned}\right\} \qquad (18)$$

On the other hand, the tangent must pass through the point B (2, 3), and so

$$C_1 + 2C_2 = 3 \qquad (19)$$

Eliminating C_1 and C_2 from (18), (19), we find $\bar{x}^2 - 4\bar{x} + 3 = 0$, whence $\bar{x}_1 = 1$ and $\bar{x}_2 = +3$. The second value of $\bar{x}$ is not suitable, and so $\bar{x} = 1$. From this we have $C_1 = -1$, $C_2 = 2$. The desired (and only) extremal is

$$y = \begin{cases} -2x-1 & \text{if} \quad -2 \leqslant x \leqslant -1, \\ x^2 & \text{if} \quad -1 \leqslant x \leqslant 1, \\ 2x-1 & \text{if} \quad 1 \leqslant x \leqslant 2 \end{cases}$$

It clearly minimizes the functional (17).

191. Find the curves on which the following functional can attain an extremum:

$$J[y] = \int_0^{10} y'^3\,dx, \quad y(0) = 0, \quad y(10) = 0$$

subject to the condition that the admissible curves cannot pass inside the area bounded by the circle $(x-5)^2 + y^2 = 9$.

192. Among the curves joining points $A\,(a,\ y_0)$ and $B\,(b,\ y_1)$, find the one that gives an extreme value to the functional

$$J\,[y] = \int_a^b y\sqrt{1 - y^2y'^2}\,dx$$

subject to the conditions $y \geqslant 0,\ 1 - y^2y'^2 \geqslant 0$.

12. THE HAMILTON-JACOBI THEORY. THE VARIATIONAL PRINCIPLES OF MECHANICS

1°. The canonical (Hamiltonian) form of Euler's equations. Euler's equations for the functional

$$J\,[y_1,\ y_2,\ \ldots,\ y_n] = \int_{x_1}^{x_2} F\,(x,\ y_1,\ y_2,\ \ldots,\ y_n,\ y_1',\ y_2',\ \ldots,\ y_n')\,dx \qquad (1)$$

are of the form

$$F_{y_k} - \frac{d}{dx}\,(F_{y_k'}) = 0 \quad (k = 1,\ 2,\ \ldots,\ n) \qquad (2)$$

When the determinant

$$\begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} \ldots F_{y_1'y_n'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} \ldots F_{y_2'y_n'} \\ \ldots\ldots\ldots\ldots \\ F_{y_n'y_1'} & F_{y_n'y_2'} \ldots F_{y_n'y_n'} \end{vmatrix} \neq 0 \qquad (3)$$

we set

$$F_{y_k'} = p_k \quad (k = 1,\ 2,\ \ldots,\ n) \qquad (4)$$

From equations (4) we can express y_k' in terms of x, y_1, y_2, $\ldots$, y_n, p_1, p_2, $\ldots$, p_n:

$$y_k' = \varphi_k\,(x,\ y_1,\ y_2,\ \ldots,\ y_n,\ p_1,\ p_2,\ \ldots,\ p_n) \qquad (5)$$

The function H of the variables x, y_1, y_2, $\ldots$, y_n, p_1, p_2, $\ldots$, p_n which is defined by the equation

$$H = [-F\,(x,\ y_1,\ y_2,\ \ldots,\ y_n,\ y_1',\ y_2',\ \ldots,\ y_n') + \sum_{k=1}^{n} y_k'F_{y'_k}\,(x,\ y_1,\ \ldots,\ y_n,\ y_1',\ \ldots,\ y_n')]\,\big|_{y_k' = \varphi_k} \qquad (6)$$

is called the *Hamiltonian of the functional* (1).

The Hamiltonian satisfies the following relations:

$$\frac{\partial H}{\partial p_k} = \frac{dy_k}{dx}, \quad \frac{\partial H}{\partial y_k} = -\frac{dp_k}{dx} \quad (k = 1, 2, \ldots, n) \qquad (7)$$

Equations (7) are known as the *canonical*, or *Hamiltonian, system of Euler's equations* (2). The variables $y_1, y_2, \ldots, y_n$, $p_1, p_2, \ldots, p_n$ are called *canonical variables.*

Note 1. The condition (3) for the functional $J[y] = \int_{x_1}^{x_2} F(x, y, y')\,dx$ yields $F_{y'y'} \neq 0$ on $[x_1, x_2]$.

Note 2. Equations (4) are, generally speaking, not unambiguously solvable for y'_k throughout the whole interval $[x_1, x_2]$. We have a local unambiguous solvability of equations (4) if the conditions of the existence theorem of the implicit function hold.

Example 1. Form the canonical system of Euler's equations for the functional

$$J[y_1, y_2] = \int_0^{\pi} (2y_1y_2 - 2y_1^2 + y_1'^2 - y_2'^2)\,dx$$

Solution. In our case

$$F = F(x, y_1, y_2, y_1', y_2') = 2y_1y_2 - 2y_1^2 + y_1'^2 - y_2'^2$$

We set

$$F_{y_1'} = p_1, \quad F_{y_2'} = p_2$$

Then

$$p_1 = 2y_1', \quad p_2 = -2y_2'$$

Here the determinant

$$\begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 \neq 0$$

Solving the resulting relations for y_1', y_2', we get

$$y_1' = \frac{p_1}{2}, \quad y_2' = -\frac{p_2}{2}$$

We find the Hamiltonian of the given functional:

$$H = (-F + y_1' F_{y_1'} + y_2' F_{y_2'})\Big|_{\substack{y_1' = \frac{p_1}{2}, \\ y_2' = -\frac{p_2}{2}}}$$

$$= (-2y_1 y_2 + 2y_1^2 + y_1'^2 - y_2'^2)\Big|_{\substack{y_1' = \frac{p_1}{2}, \\ y_2' = -\frac{p_2}{2}}}$$

$$= 2y_1^2 - 2y_1 y_2 + \frac{p_1^2}{4} - \frac{p_2^2}{4}$$

Utilizing relations (7), we get the canonical system of Euler's equations:

$$\left.\begin{aligned} \frac{dy_1}{dx} &= \frac{p_1}{2}, & \frac{dy_2}{dx} &= -\frac{p_2}{2}, \\ \frac{dp_1}{dx} &= -4y_1 + 2y_2, & \frac{dp_2}{dx} &= 2y_1 \end{aligned}\right\}$$

Here, $y_1 = y_1(x)$, $y_2 = y_2(x)$, $p_1 = p_1(x)$, $p_2 = p_2(x)$ are unknown functions of x.

Example 2. Form the canonical system of Euler's equations for the functional

$$J[y_1, y_2] = \int y_1^2 y_2^2 (x^2 + y_1' + y_2')\, dx$$

Solution. Here

$$F = y_1^2 y_2^2 (x^2 + y_1' + y_2')$$

We find the partial derivatives

$$F_{y_1'} = y_1^2 y_2^2, \quad F_{y_2'} = y_1^2 y_2^2$$

We set

$$p_1 = y_1^2 y_2^2, \qquad p_2 = y_1^2 y_2^2$$

These relations do not contain the derivatives y_1', y_2' of the unknown functions y_1 and y_2 and therefore y_1' and y_2' cannot be expressed in terms of the variables p_1 and p_2. Hence, it is impossible to form the Hamiltonian of this functional. In this example, condition (3) is not fulfilled:

$$\begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \equiv 0$$

Example 3. Form the canonical system of Euler's equations for the functional

$$J[y] = \int xyy'^3\,dx$$

Solution. We have

$$F = xyy'^3, \quad F_{y'} = 3xyy'^2$$

Set $p = 3xyy'^2$, whence

$$y' = -\sqrt{\frac{p}{3xy}}, \quad y' = \sqrt{\frac{p}{3xy}}$$

The given functional has two Hamiltonians:

$$H_1 = (-F + y'F_{y'})\Big|_{y'=-\sqrt{\frac{p}{3xy}}} = 2xyy'^3\Big|_{y'=-\sqrt{\frac{p}{3xy}}} = -\frac{2}{3\sqrt{3}}\sqrt{\frac{p^3}{xy}},$$

$$H_2 = (-F + y'F_{y'})\Big|_{y'=\sqrt{\frac{p}{3xy}}} = \frac{2}{3\sqrt{3}}\sqrt{\frac{p^3}{xy}}$$

Accordingly, we obtain two canonical systems of Euler's equations:

$$\left.\begin{aligned} \frac{dy}{dx} &= \frac{\partial H_1}{\partial p} = -\sqrt{\frac{p}{3xy}}, \\ \frac{dp}{dx} &= -\frac{\partial H_1}{\partial y} = -\frac{1}{3}\sqrt{\frac{p^3}{3xy^3}}, \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{dy}{dx} &= \sqrt{\frac{p}{3xy}}, \\ \frac{dp}{dx} &= \frac{1}{3}\sqrt{\frac{p^3}{3xy^3}} \end{aligned}\right\}$$

Form canonical systems of Euler's equations for the following functionals.

193. $J = \int xy\sqrt{y'}\,dx.$

194. $J = \int xyy'^2\,dx.$

195. $J = \int \sqrt{x^2 + y^2}\sqrt{1 + y'^2}\,dx.$

196. $J=\int_1^2 (y_1'^2+y_2^2+y_2'^2)\,dx.$

197. $J=\int (x^2+y_1y_1'^2+y_2y_2'^2)\,dx.$

198. $J=\int \left(2xy_1-y_1'^2+\frac{y_2'^3}{3}\right) dx.$

2°. The Hamilton-Jacobi equation. Jacobi's theorem. The canonical system of Euler's equations (7) is a system of Euler's equations for the functional

$$J=\int_{x_1}^{x_2}\left[\sum_{k=1}^{n} p_k y_k' - H(x,\ y_1,\ \ldots,\ y_n,\ p_1,\ \ldots,\ p_n)\right] dx$$

if $y_1, \ldots, y_n, p_1, \ldots, p_n$ are regarded as unknown functions of x.

The given functional J is a solution of a first-order partial differential equation of the form

$$\frac{\partial W}{\partial x}+H\left(x,\ y_1,\ \ldots,\ y_n,\ \frac{\partial W}{\partial y_1},\ \frac{\partial W}{\partial y_2},\ \ldots,\ \frac{\partial W}{\partial y_n}\right)=0$$

which is called the *Hamilton-Jacobi equation.*

Jacobi's theorem. *Let W be the complete integral* (see [9]) *of the Hamilton-Jacobi equation. This integral satisfies the condition*

$$\begin{vmatrix} \frac{\partial^2 W}{\partial y_1\,\partial C_1} & \frac{\partial^2 W}{\partial y_1\,\partial C_2} & \cdots & \frac{\partial^2 W}{\partial y_1\,\partial C_n} \\ \frac{\partial^2 W}{\partial y_2\,\partial C_1} & \frac{\partial^2 W}{\partial y_2\,\partial C_2} & \cdots & \frac{\partial^2 W}{\partial y_2\,\partial C_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 W}{\partial y_n\,\partial C_1} & \frac{\partial^2 W}{\partial y_n\,\partial C_2} & \cdots & \frac{\partial^2 W}{\partial y_n\,\partial C_n} \end{vmatrix} \neq 0$$

Then the equations

$$\frac{\partial W}{\partial C_k}=B_k,\quad \frac{\partial W}{\partial y_k}=p_k \quad (k=1,\ 2,\ \ldots,\ n)$$

where C_k and B_k are arbitrary constants, yield a solution of the canonical system (7), *which solution depends on $2n$ arbitrary constants.*

Example 4. Find the extremals of the functional

$$J = \int_{x_1}^{x_2} \sqrt{x^2 + y^2}\sqrt{1 + y'^2}\, dx$$

by using the solution of the Hamilton-Jacobi equation.

Solution. To obtain the Hamilton-Jacobi equation, we find the Hamiltonian of the given functional, which is

$$H = -\sqrt{x^2 + y^2 - p^2}$$

The Hamilton-Jacobi equation is of the form

$$\frac{\partial W}{\partial x} - \sqrt{x^2 + y^2 - \left(\frac{\partial W}{\partial y}\right)^2} = 0$$

or

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = x^2 + y^2 \tag{8}$$

Let us rewrite (8) as

$$\left(\frac{\partial W}{\partial x}\right)^2 - x^2 + \left(\frac{\partial W}{\partial y}\right)^2 - y^2 = 0$$

and apply the method of separation of variables. It is clear that if we demand that

$$\left(\frac{\partial W}{\partial x}\right)^2 - x^2 = -C, \quad \left(\frac{\partial W}{\partial y}\right)^2 - y^2 = C$$

where C is an arbitrary constant, then (8) will be satisfied. From this we find

$$\frac{\partial W}{\partial x} = \sqrt{x^2 - C}, \quad \frac{\partial W}{\partial y} = \sqrt{y^2 + C}$$

The complete integral of equation (8) is

$$W = \int \sqrt{x^2 - C}\, dx + \int \sqrt{y^2 + C}\, dy$$

$$= \frac{1}{2} x \sqrt{x^2 - C} - \frac{C}{2} \ln\left|x + \sqrt{x^2 - C}\right| + \frac{1}{2} y \sqrt{y^2 + C}$$

$$+ \frac{C}{2} \ln\left|y + \sqrt{y^2 + C}\right| + C_0$$

where C and C_0 are arbitrary constants.

The general solution of Euler's equation of the given functional can be found from the relation $\frac{\partial W}{\partial C} = \frac{\widetilde{A}}{2}$, where $\widetilde{A}$ is an arbitrary constant. We have

$$-\frac{x}{4\sqrt{x^2-C}} - \frac{1}{2}\ln|x+\sqrt{x^2-C}| + \frac{C}{4}\frac{1}{(x+\sqrt{x^2-C})\sqrt{x^2-C}} + \frac{1}{4}\frac{y}{\sqrt{y^2+C}} + \frac{1}{2}\ln|y+\sqrt{y^2+C}| + \frac{C}{4}\frac{1}{(y+\sqrt{y^2+C})\sqrt{y^2+C}} = \frac{\widetilde{A}}{2}$$

Simplifying, we get

$$\ln\left|\frac{y+\sqrt{y^2+C}}{x+\sqrt{x^2-C}}\right| = \widetilde{A} \quad \text{or} \quad \frac{y+\sqrt{y^2+C}}{x+\sqrt{x^2-C}} = A \quad (A = \pm e^{\widetilde{A}})$$

whence, finally, we obtain

$$x^2 - \frac{1-A^2}{A}xy - y^2 = C\left(\frac{A^2+1}{2A}\right)^2$$

This is a family of hyperbolas.

Find the extremals of the following functionals.

199. $J = \int_{x_1}^{x_2} xy\sqrt{y'}\,dx.$

200. $J = \int_{1}^{e} xyy'^2\,dx;\ y(1) = 0,\ y(e) = 1.$

201. $J = \int_{x_1}^{x_2} G(y)\sqrt{1+y'^2}\,dx.$

202. Find the minimum of the functional

$$J = \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)dx$$

if the values at the ends of the interval are not given.

203. Find the function of the field $p(x, y)$ and the field of extremals itself (the extremals pass through the origin)

of the functional

$$J = \int_{(0,\,0)}^{(x,\,y)} \frac{\sqrt{1+y'^2}}{y}\,dx \quad (y>0)$$

204. Among the curves joining the straight line $x = 0$ to the point $M_1(x_1, y_1)$, where $x_1 > 0$, $y_1 > 0$, find the one that minimizes the functional

$$J = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx \quad (y>0)$$

Suppose we have a functional

$$J[y] = \int_{x_1}^{x_2} F(x,\ y,\ y')\,dx$$

and its field of extremals $y = \varphi(x, C)$ is given. Then at every point of the field we know the direction of the transversal of the field passing through that point. All transversals of the field are obtained as solutions to a first-order differential equation:

$$F_{y'}[x,\ \varphi(x,\ C),\ \varphi'_x(x,\ C)]\frac{dy}{dx} = H[x,\ \varphi(x,\ C),\ \varphi'_x(x,\ C)]$$

where in place of the parameter C that determines the extremals of the field we have to substitute its expression in terms of the coordinates of the points of the field. Here, $H(x, y, p)$ is the Hamiltonian.

Example 5. Find the transversals for the field of extremals $y = Cx$ of the functional

$$J = \int_{x_1}^{x_2} y'^2\,dx$$

Solution. Let us find the Hamiltonian of the given functional. We have

$$F = y'^2, \quad F_{y'} = 2y' \quad (F_{y'y'} = 2 \neq 0)$$

Setting $p = F_{y'}$, we find $y' = \frac{p}{2}$. Then

$$H = (-y'^2 + 2y' \cdot y')\Big|_{y'=\frac{p}{2}} = \frac{p^2}{4}$$

We obtain the transversals by solving the differential equation

$$F_{y'}\Big|_{y=Cx}\frac{dy}{dx}=H\Big|_{p=2y'=2C}$$

where in place of C we have to substitute its expression in terms of the coordinates of the points of the field $C=\frac{y}{x}$. We get

$$2y'\Big|_{y=Cx}\frac{dy}{dx}=\frac{p^2}{4}\Big|_{p=2C}\quad\text{or}\quad 2C\frac{dy}{dx}=C^2$$

Since $C\neq 0$, it follows that $2\frac{dy}{dx}=C$ or $2\frac{dy}{dx}=\frac{y}{x}$, whence we find that the family of transversals $y^2=\widetilde{C}x$ are parabolas.

205. Find the transversals for the field of extremals $y=Cx$ of the functional

$$J=\int_{x_1}^{x_2}F(y')\,dx$$

206. Find the transversals for the field of extremals $y=x+C$ of the functional

$$J=\int_{x_1}^{x_2}(xy'^4-2yy'^3)\,dx$$

207. Find the transversals for the field of extremals $y=x-\frac{x^2}{C}$ of the functional

$$J=\int_{x_1}^{x_2}\sqrt{y(1-y'^2)}\,dx\quad(C>0,\ x>0,\ y\geqslant 0)$$

Knowing the Hamilton-Jacobi equation

$$\frac{\partial W}{\partial x}+H\left(x,\ y,\ \frac{\partial W}{\partial y}\right)=0$$

for the functional

$$J=\int_{x_1}^{x_2}F(x,\ y,\ y')\,dx$$

we can restore the integrand $F(x, y, y')$, which is a solution of the first-order differential equation

$$F - zF'_z = -H(x, y, F'_z) \tag{9}$$

where $H(x, y, p)$ is the Hamiltonian of the desired functional; $F(x, y, z)$ is the desired function (x and y are regarded as parameters). After finding $F(x, y, z)$, substitute the derivative y' in place of z.

Note. Equation (9) is Clairaut's equation. As a rule, the general solution of Clairaut's equation is discarded since in this case the integrand $F(x, y, y')$ is linear in y' and the variational problem is not always solvable (see Sec. 4). Therefore, we take only a particular solution of Clairaut's equation, which is the desired function $F(x, y, z)$.

Example 6. The Hamilton-Jacobi equation for the problem of the extremum of the functional $J = \int_{x_1}^{x} F(x, y, y')\, dx$ is of the form

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = x^2 + y^2$$

Find the function $F(x, y, y')$.

Solution. Solving the given equation for the derivative $\frac{\partial W}{\partial x}$, we have

$$\frac{\partial W}{\partial x} = \sqrt{x^2 + y^2 - \left(\frac{\partial W}{\partial y}\right)}$$

or

$$\frac{\partial W}{\partial x} - \sqrt{x^2 + y^2 - \left(\frac{\partial W}{\partial y}\right)^2} = 0$$

Hence, the Hamiltonian is

$$H = -\sqrt{x^2 + y^2 - p^2}$$

Equation (9) for finding the desired function F is of the form

$$F - z\frac{dF}{dz} = \sqrt{x^2 + y^2 - \left(\frac{dF}{dz}\right)^2} \tag{10}$$

Differentiate both sides of (10) with respect to z:

$$\frac{dF}{dz} - \frac{dF}{dz} - z\frac{d^2F}{dz^2} = -\frac{\frac{dF}{dz}\frac{d^2F}{dz^2}}{\sqrt{x^2+y^2-\left(\frac{dF}{dz}\right)^2}}$$

Discarding the case where $\frac{d^2F}{dz^2}=0$ (it yields the general solution), we get

$$z = \frac{\frac{dF}{dz}}{\sqrt{x^2+y^2-\left(\frac{dF}{dz}\right)^2}}$$

Solving this relation for the derivative $\frac{dF}{dz}$, we obtain

$$\frac{dF}{dz} = \frac{z\sqrt{x^2+y^2}}{\sqrt{1+z^2}} \tag{11}$$

Putting (11) into (10), we get

$$F = z\frac{z\sqrt{x^2+y^2}}{\sqrt{1+z^2}} + \sqrt{x^2+y^2-\frac{z^2(x^2+y^2)}{\sqrt{1+z^2}}}$$

$$= \sqrt{x^2+y^2}\sqrt{1+z^2}$$

Thus, the desired integrand has the form

$$F = \sqrt{x^2+y^2}\sqrt{1+y'^2}$$

In the following examples, find the functionals if their Hamilton-Jacobi equations are known.

208. $\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = 1.$

209. $4\frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial y} = x^2y^2.$

210. $4xy\frac{\partial W}{\partial x} + \left(\frac{\partial W}{\partial y}\right)^2 = 0.$

211. $\left(x\frac{\partial W}{\partial x}\right)^2 + \left(y\frac{\partial W}{\partial y}\right)^2 = x^2+y^2.$

3°. The variational principles of mechanics.

(a) *The Hamilton-Ostrogradsky principle.* Let there be a system of n material points $M_k(x_k, y_k, z_k)$ $(k = 1,$

2, . . ., n) with masses, respectively, m_k ($k = 1, 2, \ldots, n$). We assume the motion of the system is subject to the constraints

$$\varphi_j(x, y, z, t) = 0 \quad (j = 1, 2, \ldots, m, \text{ where } m \leqslant n) \tag{12}$$

and occurs under the action of the forces $F_k(X_k, Y_k, Z_k)$ ($k = 1, 2, \ldots, n$) having the potential $U = U(x_k, y_k, z_k, t)$:

$$X_k = \frac{\partial U}{\partial x_k}, \quad Y_k = \frac{\partial U}{\partial y_k}, \quad Z_k = \frac{\partial U}{\partial z_k}$$

The kinetic energy of this system will then be equal to

$$T = \frac{1}{2} \sum_{k=1}^{n} m_k (\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2)$$

Suppose that the system moves from some position A at time $t = t_0$ to a position B at time $t = t_1$. Of all possible motions of the system from A to B, a class of admissible motions is chosen; namely, those motions which are compatible with the given constraints and which, in the given interval of time $[t_0, t_1]$, translate the system from A to B.

The actual motion of the system from position A to position B satisfies the necessary condition $\delta J = 0$ for the extremum of the integral

$$J = \int_{t_0}^{t_1} (T + U)\, dt \tag{13}$$

To each admissible motion of the system there correspond $3n$ functions $x_k(t)$, $y_k(t)$, $z_k(t)$ ($k = 1, 2, \ldots, n$) that are defined in the interval $[t_0, t_1]$, satisfy the equations (12), and have specified values at the ends of the interval $[t_0, t_1]$. We thus have a variational problem with constraints (12) and fixed boundaries.

To solve this problem we set up the auxiliary Lagrangian function

$$F = T + U + \sum_{j=1}^{m} \lambda_j(t)\, \varphi_j$$

For this function we write out the system of Euler-Ostrogradsky equations:

$$\left.\begin{aligned} m_k\ddot{x}_k - X_k - \sum_{j=1}^{m} \lambda_j(t)\frac{\partial \varphi_j}{\partial x_k} = 0, \\ m_k\ddot{y}_k - Y_k - \sum_{j=1}^{m} \lambda_j(t)\frac{\partial \varphi_j}{\partial y_k} = 0, \\ m_k\ddot{z}_k - Z_k - \sum_{j=1}^{m} \lambda_j(t)\frac{\partial \varphi_j}{\partial z_k} = 0 \end{aligned}\right\} \qquad (14)$$

The equations (14) coincide with the differential equations of the actual motion of the system.

(b) *Principle of least action in the form of Lagrange.* Suppose the constraints φ_j and the potential U do not depend on the time t. We then have the energy integral $T - U = h = \text{const}$. The integral

$$J = \int_{t_0}^{t_1} T\,dt$$

is called the *action*. From the integral (13) it follows that

$$\int_{t_0}^{t_1} (T+U)\,dt = 2\int_{t_0}^{t_1} T\,dt - \int_{t_0}^{t_1} h\,dt$$

By the Hamilton-Ostrogradsky principle, the action integral for the actual motion must have a minimal value:

$$J = \int_{t_0}^{t_1} T\,dt = \min$$

The principle of least action may be cast in the form of Jacobi:

$$\int_{\gamma} \sqrt{2(U+h)}\,ds = \min$$

(ds is the differential of the arc γ) in which the time is eliminated.

Note 1. Here, the admissible motions are those that satisfy the equations of constraint $\varphi_j(x, y, z) = 0$ $(j = 1, 2, \ldots, m)$ and the equation $T - U = h$ with the same value of h as for the actual motion, and that have fixed initial and terminal positions and a fixed initial instant of time t_0. The terminal time is not fixed.

Note 2. The potential energy does not enter into the integral but into the accessory condition $T - U = h$. Form the auxiliary Lagrangian function

$$F = \frac{1}{2} T + \frac{1}{2} (U + h) + \sum_{j=1}^{m} \lambda_j \varphi_j$$

Then write down the Euler-Ostrogradsky equations of our problem, which are equations of actual motion:

$$m_k \ddot{x}_k = \frac{\partial U}{\partial x_k} + 2 \sum_{j=1}^{m} \lambda_j \frac{\partial \varphi_j}{\partial x_k},$$

$$m_k \ddot{y}_k = \frac{\partial U}{\partial y_k} + 2 \sum_{j=1}^{m} \lambda_j \frac{\partial \varphi_j}{\partial y_k},$$

$$m_k \ddot{z}_k = \frac{\partial U}{\partial z_k} + 2 \sum_{j=1}^{m} \lambda_j \frac{\partial \varphi_j}{\partial z_k}$$

Example 7. By proceeding from the principle of least action, find the trajectory of a material point (of unit mass) moving under the action of gravity.

Solution. Direct the y-axis upwards. Then the potential of the force of gravity will be

$$U = -gy \tag{15}$$

According to the principle of least action, the integral

$$J = \int_{\gamma} \sqrt{2(U + h)}\, ds \tag{16}$$

must have a minimal value on the desired trajectory γ. Hence, the trajectory of the point will be an extremal of

the functional (16). Substituting (15) into (16), we get

$$J = \int_{x_1}^{x_2} \sqrt{2(h-gy)}\sqrt{1+y'^2}\,dx \qquad (17)$$

The Hamilton-Jacobi equation is of the form

$$\frac{\partial W}{\partial x} - \sqrt{2h - 2gy - \left(\frac{\partial W}{\partial y}\right)^2} = 0$$

or

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = 2(h-gy)$$

Its complete integral is

$$W = Ax + \int \sqrt{2h - 2gy - A^2}\,dy$$

$$= Ax - \frac{1}{3g}(2h - 2gy - A^2)^{3/2} + B$$

where A and B are arbitrary constants.

Let us find the extremals of the functional (17):

$$x + \frac{A}{g}(2h - 2gy - A^2)^{1/2} = C$$

or

$$y = \frac{h}{g} - \frac{A^2}{2g} - \frac{g}{2A^2}(x-C)^2, \ A \text{ and } C \text{ are constant}$$

In particular, the extremals passing through the origin are found from the condition $y(0) = 0$. We obtain a one-parameter family of parabolas:

$$y = -\frac{g}{2A^2}x^2 + \frac{\sqrt{2h-A^2}}{A}x$$

212. By proceeding from the action integral

$$J = \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx \quad (y>0)$$

find the trajectory of motion of a point in a plane under the action of a repulsive force acting from the x-axis, the force being proportional to the distance of the point to this straight line and directed parallel to the y-axis, subject to the condition that $v^2 - y^2 = 0$.

213. A material point describes a circle $\rho = 2R\cos\varphi$ (ρ, φ are polar coordinates) of radius R under the action of a central force $\frac{k}{\rho^5}$ that is inversely proportional to the fifth power of the distance from the centre, which is located at the origin. Show that the action integral attains a strong minimum on any arc of this circle $\left(-\frac{\pi}{2} < \varphi_1 \leqslant \varphi \leqslant \varphi_2 < \frac{\pi}{2}\right)$.

214. Study the motion of a material point under the action of an attractive central force proportional to the distance from the centre O by proceeding from the principle of least action and by applying the Hamilton-Jacobi method.

CHAPTER III

DIRECT METHODS IN THE CALCULUS OF VARIATIONS

13. EULER'S FINITE-DIFFERENCE METHOD

Suppose we have an elementary variational problem: to find the extremum of the functional

$$J[y(x)] = \int_a^b F(x, y, y')\,dx;\ y(a) = A,\ y(b) = B \qquad (1)$$

In Euler's method, the values of the functional (1) are considered not on arbitrary curves admissible in the given variational problem, but only on polygonal curves composed of a given number n of straight-line segments with specified abscissas of the vertices:

$$a + \Delta x,\ a + 2\Delta x,\ \ldots,\ a + (n-1)\,\Delta x,\ \text{where } \Delta x = \frac{b-a}{n}$$

On these polygonal curves, the functional $J[y(x)]$ becomes a function $\Phi(y_1, y_2, \ldots, y_{n-1})$ of the ordinates $y_1, y_2, \ldots, y_{n-1}$ of the vertices of the polygonal curve. The ordinates $y_1, y_2, \ldots, y_{n-1}$ are chosen so that the function $\Phi(y_1, y_2, \ldots, y_{n-1})$ attains an extremum; that is, they are determined from the system of equations

$$\frac{\partial \Phi}{\partial y_1} = 0,\ \frac{\partial \Phi}{\partial y_2} = 0,\ \ldots,\ \frac{\partial \Phi}{\partial y_{n-1}} = 0 \qquad (2)$$

The resulting polygonal curve is an approximate solution of the variational problem (1).

Example. Find an approximate solution of the problem of the minimum of the functional

$$J[y(x)] = \int_0^1 (y'^2 + 2y)\,dx, \quad y(0) = y(1) = 0$$

Solution. Take $\Delta x = \frac{1-0}{5} = 0.2$ and set

$$y_0 = y(0) = 0, \quad y_1 = y(0.2), \quad y_2 = y(0.4),$$
$$y_3 = y(0.6), \quad y_4 = y(0.8), \quad y_5 = y(1) = 0$$

The values of the derivatives are then replaced approximately by the formula

$$y'_k = y'(x_k) \approx \frac{y_{k+1} - y_k}{\Delta x}$$

Then

$$y'(0) = \frac{y_1 - 0}{0.2}, \quad y'(0.2) = \frac{y_2 - y_1}{0.2},$$

$$y'(0.4) = \frac{y_3 - y_2}{0.2},$$

$$y'(0.6) = \frac{y_4 - y_3}{0.2}, \quad y'(0.8) = \frac{0 - y_4}{0.2}$$

We replace the given integral with the sum using the formula of rectangles:

$$\int_a^b f(x)\,dx \approx [f(a) + f(x_1) + f(x_2) + \ldots + f(x_{n-1})]\,\Delta x$$

We then have

$$\Phi(y_1, y_2, y_3, y_4) = \left[\left(\frac{y_1}{0.2}\right)^2 + \left(\frac{y_2 - y_1}{0.2}\right)^2 + 2y_1 + \left(\frac{y_3 - y_2}{0.2}\right)^2 + 2y_2 + \left(\frac{y_4 - y_3}{0.2}\right)^2 + 2y_3 + \left(-\frac{y_4}{0.2}\right)^2 + 2y_4\right] \cdot 0.2$$

Now form a system of equations for determining the ordinates y_1, y_2, y_3, y_4 of the desired polygonal curve:

$$\left.\begin{aligned}
\frac{1}{0.2}\cdot\frac{\partial\Phi}{\partial y_1} &= \frac{y_1}{0.02} - \frac{y_2-y_1}{0.02} + 2 = 0,\\
\frac{1}{0.2}\cdot\frac{\partial\Phi}{\partial y_2} &= \frac{y_2-y_1}{0.02} - \frac{y_3-y_2}{0.02} + 2 = 0,\\
\frac{1}{0.2}\cdot\frac{\partial\Phi}{\partial y_3} &= \frac{y_3-y_2}{0.02} - \frac{y_4-y_3}{0.02} + 2 = 0,\\
\frac{1}{0.2}\cdot\frac{\partial\Phi}{\partial y_4} &= \frac{y_4-y_3}{0.02} + \frac{y_4}{0.02} + 2 = 0
\end{aligned}\right\}$$

or

$$\left.\begin{aligned}
2y_1 - y_2 &= -0.04,\\
-y_1 + 2y_2 - y_3 &= -0.04,\\
-y_2 + 2y_3 - y_4 &= -0.04,\\
-y_3 + 2y_4 &= -0.04
\end{aligned}\right\}$$

The solution of this system yields $y_1 = -0.08$, $y_2 = -0.12$, $y_3 = -0.12$, $y_4 = -0.08$. The values of the exact solution $y = \frac{x^2-x}{2}$ coincide, at the appropriate points, with the values of the approximate solution.

Find approximate solutions in the following problems of the minima of functionals.

215. $J[y] = \int_0^1 (y'^2 + y^2 + 2xy)\,dx; \quad y(0) = y(1) = 0$

Hint. Take $\Delta x = 0.2$.

216. $J[y] = \int_0^1 (y'^2 + 1)\,dx;$

(a) $y(0) = 0, \quad y(1) = 0,$

(b) $y(0) = 0, \quad y(1) = 1.$

14. RITZ METHOD. KANTOROVICH METHOD

1°. Ritz method. The underlying idea of this method lies in the fact that when we seek the extremum of a functional $J[y(x)]$, we do not consider the whole space of ad-

missible functions but only all possible linear combinations of admissible functions of the form

$$y_n(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x) \tag{1}$$

where α_i are constants, and the system $\{\varphi_i(x)\}$, called the *system of coordinate functions*, is such that the functions $\varphi_i(x)$ are linearly independent and form a complete system of functions in the space under consideration.

The requirement that $y_n(x)$ be admissible functions generally speaking imposes on the coordinate functions $\varphi_i(x)$ certain accessory conditions like the conditions of smoothness or satisfaction of the boundary conditions. On such linear combinations, the functional $J[y(x)]$ becomes a function of the arguments $\alpha_1, \alpha_2, \ldots, \alpha_n$:

$$J[y_n(x)] = \Phi(\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{2}$$

Let us find the values of $\alpha_1, \alpha_2, \ldots, \alpha_n$ that make the function $\Phi(\alpha_1, \alpha_2, \ldots, \alpha_n)$ an extremum. To do this, we solve a system of equations that, generally speaking, are nonlinear in $\alpha_1, \alpha_2, \ldots, \alpha_n$:

$$\frac{\partial \Phi}{\partial \alpha_i} = 0 \quad (i = 1, 2, \ldots, n) \tag{3}$$

and then substitute the values of α_i thus found into (1). The resulting sequence $\{y_n(x)\}$ is a minimizing sequence, that is, a sequence for which the sequence of values of the functional $\{J[y_n(x)]\}$ converges to a minimum or to the lower bound of values of the functional $J[y]$. However,

$$\lim_{n\to\infty} J[y_n(x)] = \min J[y(x)]$$

does not yet imply that $\lim_{n\to\infty} y_n(x) = y(x)$. The minimizing sequence may not tend to the function that gives an extremum in the class of admissible functions.

It is possible to indicate conditions that ensure the existence of an absolute minimum of the functional and its attainment on the functions $\{y_n(x)\}$.

When the extremum of the functional

$$J[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx,$$

$$y(x_1) = y_1, \quad y(x_2) = y_2$$

is sought, these conditions are:

1. The function $F(x, y, z)$ is continuous with respect to all the arguments for any z and for $(x, y) \in D$, where D is a closed domain of the XY-plane in which lie the curves $y_n(x)$.

2. There exist constants $\alpha > 0$, $p > 1$, β for which

$$F(x, y, z) \geqslant \alpha |z|^p + \beta$$

no matter what z is and for any point $(x, y) \in D$.

3. The function $F(x, y, z)$ has a continuous partial derivative $F_z(x, y, z)$; for any point $(x, y) \in D$, this derivative is a nondecreasing function of z $(-\infty < z < +\infty)$.

The foregoing conditions are, in particular, fulfilled for functionals of the form

$$J[y] = \int_{x_1}^{x_2} [p(x) y'^2 + q(x) y^2 + 2r(x) y]\,dx,$$

$$y(x_1) = a, \quad y(x_2) = b$$

where $p(x)$, $q(x)$, $r(x)$ are given functions continuous on $[x_1, x_2]$, $p(x)$ has a continuous derivative $p'(x)$, and $p(x) > 0$, $q(x) \geqslant 0$.

If this method is used to determine the absolute extremum of a functional, then the approximate value of the minimum of the functional is obtained with an excess and that of the maximum, with a deficit. The successful use of this method depends largely on how apt the choice is of the system of coordinate functions $\{\varphi_i(x)\}$.

In many cases it suffices to take a linear combination of two or three functions $\varphi_i(x)$ in order to obtain quite satisfactory an approximation to the exact solution.

In finding the approximate extremum of functionals $J[z(x_1, x_2, \ldots, x_n)]$ that depend on functions of several independent variables, a coordinate system of functions is

chosen:

$$\varphi_1(x_1, x_2, \ldots, x_n), \quad \varphi_2(x_1, x_2, \ldots, x_n),$$
$$\ldots, \varphi_m(x_1, x_2, \ldots, x_n), \ldots$$

An approximate solution of the variational problem is sought in the form

$$z_m = \sum_{k=1}^{m} \alpha_k \varphi_k(x_1, x_2, \ldots, x_n)$$

where the coefficients α_k are certain constant numbers. To determine them, we proceed as before and form a system of equations $\frac{\partial \Phi}{\partial \alpha_k} = 0$ $(k = 1, 2, \ldots, n)$, where $\Phi(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the result of substituting z_m into the functional $J[z]$.

Example 1. Find an approximate solution to the problem of the minimum of the functional

$$J[y] = \int_0^1 (y'^2 - y^2 + 2xy)\, dx, \tag{4}$$

$$y(0) = y(1) = 0 \tag{5}$$

and compare it with the exact solution.

Solution. We choose the system of coordinate functions $\varphi_k(x)$ thus:

$$\varphi_k(x) = (1 - x)\, x^k \quad (k = 1, 2, \ldots)$$

The functions $\varphi_k(x)$ clearly satisfy the boundary conditions $\varphi_k(0) = \varphi_k(1) = 0$, are linearly independent, and constitute a complete system in the space $C_1[0, 1]$.

For $k = 1$ we get $y_1(x) = \alpha_1(x - x^2)$. Substituting this expression for $y_1(x)$ into the functional (4), we get

$$J[y_1(x)] = \int_0^1 [\alpha_1^2 (1 - 2x)^2 - \alpha_1^2 (x - x^2)^2 + 2\alpha_1 x (x - x^2)]\, dx$$

$$= \int_0^1 [\alpha_1^2 (1 - 4x + 4x^2 - x^2 + 2x^3 - x^4) + 2\alpha_1 (x^2 - x^3)]\, dx$$

$$= \frac{3}{10}\alpha_1^2 + \frac{1}{6}\alpha_1 = \Phi(\alpha_1)$$

The coefficient α_1 is found from the equation

$$\frac{\partial \Phi}{\partial \alpha_1} = \frac{3}{5}\alpha_1 + \frac{1}{6} = 0$$

whence $\alpha_1 = -\frac{5}{18}$. Consequently

$$y_1(x) = -\frac{5}{18}x + \frac{5}{18}x^2$$

Exact solution. Euler's equation for the given functional is

$$y'' + y = x$$

Solving this inhomogeneous linear equation, we find

$$y(x) = C_1 \cos x + C_2 \sin x + x$$

Using the boundary conditions $y(0) = y(1) = 0$, we finally get

$$y(x) = x - \frac{\sin x}{\sin 1}$$

Let us compare the exact solution with the approximate one:

x	Exact solution	Approximate solution
0.00	0	0
0.25	−0.044	−0.052
0.50	−0.070	−0.069
0.75	−0.060	−0.052
1.00	0	0

Example 2. Find an approximate solution of the nonlinear equation

$$y'' = \frac{3}{2}y^2$$

that satisfies the conditions $y(0) = 4$, $y(1) = 1$.

Solution. To this boundary-value problem there corresponds the variational problem

$$J[y] = \int_0^1 (y'^2 + y^3)\,dx; \quad y(0) = 4, \quad y(1) = 1$$

We will seek the solution in the form

$$y_1(x) = 4 - 3x + \alpha_1 (x - x^2)$$

where for any value of α_1 it is clear that $y_1(x)$ satisfies the given boundary conditions.

We have

$$J[y_1] = \int_0^1 \{[\alpha_1 (1-2x) - 3]^2 + [4 - 3x + \alpha_1 (x - x^2)]^3\}\, dx$$

whence

$$\frac{\partial J[y_1]}{\partial \alpha_1} = \int_0^1 \{(1-2x) \cdot 2[\alpha_1 (1-2x) - 3] + 3(x - x^2)[4 - 3x + \alpha_1 (x - x^2)]^2\}\, dx$$

The condition $\frac{\partial J[y_1]}{\partial \alpha_1} = 0$ assumes the form

$$9\alpha_1^2 + 490\alpha_1 + 1407 = 0$$

so that for $\alpha_1 = -3.0413$ we everywhere get a positive solution of the problem:

$$y_1(x) = 3.0413\, x^2 - 6.0413\, x + 4$$

Find approximate solutions of the following problems of the minimum of a functional. Compare with the exact solutions.

217. $J[y(x)] = \int_0^1 (y'^2 + 2y)\, dx;\ y(0) = y(1) = 0.$

218. $J[y(x)] = \int_0^2 (2xy + y^2 + y'^2)\, dx;\ y(0) = y(2) = 0.$

219. Find an approximate solution to the problem of minimizing the integral

$$J[y(x)] = \int_{-1}^1 (y'^2 - k^2 y^2)\, dx; \quad y(-1) = y(1) = 0$$

subject to the accessory condition $\int_{-1}^1 y^2(x)\, dx = 1.$

Example 3. Find an approximate solution of the problem of extremizing the functional

$$J[z(x, y)] = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 - 2z \right] dx\, dy$$

where D is a square, $-a \leqslant x \leqslant a$, $-a \leqslant y \leqslant a$, and $z = 0$ on the boundary of the square.

Solution. We will seek the approximate solution in the form

$$z_0(x, y) = \alpha_0 (x^2 - a^2)(y^2 - a^2)$$

It is clear that the function $z_0(x, y)$ thus constructed satisfies the stated boundary conditions. Substituting $z_0(x, y)$, $z'_{0x}(x, y)$ and $z'_{0y}(x, y)$ into the functional, we get, after integrating,

$$J[z_0] = \frac{256}{45} \alpha_0^2 a^8 - \frac{32}{9} \alpha_0 a^6 = \Phi(\alpha_0)$$

Furthermore,

$$\frac{\partial \Phi}{\partial \alpha_0} = \frac{512}{45} \alpha_0 a^8 - \frac{32}{9} a^6 = 0$$

whence $\alpha_0 = \frac{5}{16a^2}$ so that $z_0(x, y) = \frac{5}{16a^2}(x^2 - a^2)(y^2 - a^2)$.

220. Find an approximate solution to the problem of the extremum of the functional

$$J[z(x, y)] = \iint_D \left[\left(\frac{\partial z}{\partial x} - y\right)^2 + \left(\frac{\partial z}{\partial y} + x\right)^2 \right] dx\, dy$$

where D is a domain bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

221. Find an approximate solution $z_3(x, y)$ to the problem of minimizing the functional

$$J[z(x, y)] = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right] dx\, dy$$

where the domain D is $x > 0$, $y > 0$, $x + y < 1$ and the function $z(x, y)$ satisfies the condition $z|_\Gamma = x^2 + y^2$ on the boundary Γ: $x = 0$, $y = 0$, $x + y = 1$.

2°. Kantorovich method. This method occupies an intermediate position between the exact solution of a problem

and the Ritz method and is used for testing for an extremum functionals

$$J\,[z\,(x_1,\ x_2,\ \ldots,\ x_n)] \tag{6}$$

that are dependent on functions of several independent variables ($n \geqslant 2$). As in the case of the Ritz method, we choose a coordinate system of functions $\{\varphi_k\,(x_1,\ x_2,\ \ldots\ \ldots,\ x_n)\}$ and seek an approximate solution in the form

$$z_m = \sum_{k=1}^{m} \alpha_k\,(x_j)\,\varphi_k\,(x_1,\ x_2,\ \ldots,\ x_n) \tag{7}$$

but now the coefficients $\alpha_k\,(x_j)$ are the unknown functions of one of the independent variables.

On functions of the form (7), the functional (6) is converted into the functional $\widetilde{J}\,[\alpha_1\,(x_j),\ \alpha_2\,(x_j),\ \ldots,\ \alpha_m\,(x_j)]$ dependent on m functions $\alpha_1\,(x_j),\ \alpha_2\,(x_j),\ \ldots,\ \alpha_m\,(x_j)$. These functions are chosen so that the functional $\widetilde{J}$ attains an extremum and are determined from the necessary conditions for the extremum of the functional $\widetilde{J}$.

Using the Kantorovich method, we obtain an approximate solution, which, generally speaking, is more exact than in the Ritz method for the same coordinate functions $\{\varphi_k\,(x_1,\ x_2,\ \ldots,\ x_n)\}$ and with the same number m of terms in the approximation.

Example 4. Find an approximate solution of the Poisson equation

$$\Delta z = -1 \text{ in the rectangle } D: \begin{cases} -a \leqslant x \leqslant a, \\ -b \leqslant y \leqslant b \end{cases} \tag{8}$$

subject to the condition $z = 0$ on the contour.

Solution. The equation $\Delta z = -1$ is the Euler-Ostrogradsky equation for the functional

$$J\,[y] = \iint\limits_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 - 2z\right] dx\,dy \tag{9}$$

We seek the solution in the form

$$z_1\,(x,\ y) = (b^2 - y^2)\,\alpha\,(x)$$

The function $z_1\,(x,\ y)$ clearly satisfies the boundary conditions (8) on the straight lines $y = \pm b$.

Substituting this value of z_1 into the functional (9), we get

$$J[z_1] = \int_{-a}^{a} \left(\frac{16}{15} b^5 \alpha'^2 + \frac{8}{3} b^3 \alpha^2 - \frac{8}{3} b^3 \alpha\right) dx \qquad (10)$$

The Euler equation for the functional (10) is

$$\alpha'' - \frac{5}{2b^2} \alpha = -\frac{5}{4b^2} \qquad (11)$$

Equation (11) is a homogeneous linear equation with constant coefficients and its general solution is

$$\alpha(x) = C_1 \cosh \sqrt{\frac{5}{2}} \frac{x}{b} + C_2 \sinh \sqrt{\frac{5}{2}} \frac{x}{b} + \frac{1}{2}$$

We find the constants C_1 and C_2 from the boundary conditions

$$\alpha(-a) = \alpha(a) = 0$$

This yields $C_2 = 0$, $C_1 = -\dfrac{1}{2 \cosh \sqrt{\frac{5}{2}} \frac{a}{b}}$ so that

$$\alpha(x) = \frac{1}{2}\left[1 - \frac{\cosh \sqrt{\frac{5}{2}} \frac{x}{b}}{\cosh \sqrt{\frac{5}{2}} \frac{a}{b}}\right]$$

We then have

$$z_1(x, y) = \frac{b^2 - y^2}{2}\left[1 - \frac{\cosh \sqrt{\frac{5}{2}} \frac{x}{b}}{\cosh \sqrt{\frac{5}{2}} \frac{a}{b}}\right]$$

To obtain a more exact approximation, we can seek the solution of the problem in the form

$$z_2(x, y) = (b^2 - y^2)\, \alpha_1(x) + (b^2 - y^2)^2\, \alpha_2(x)$$

222. Find an approximate solution of the Poisson equation $\Delta z = -1$ in the domain D, which is an equilateral triangle bounded by the straight lines $y = \pm \frac{\sqrt{3}}{3} x$ and $x = b$, which solution vanishes on the boundary of the domain.

223. Find an approximate solution of the equation $\Delta z = -1$ in the domain D, which is an isosceles trapezoid bounded by the straight lines $y = \pm \frac{\sqrt{3}}{3} x$ and $x = 1$, $x = 3$, which solution vanishes on the boundary of the domain.

15. VARIATIONAL METHODS FOR FINDING EIGENVALUES AND EIGENFUNCTIONS

The Sturm-Liouville equation

$$-\frac{d}{dx}(p(x)\,y') + q(x)\,y = \lambda y \tag{1}$$

where $p(x) > 0$ has a continuous derivative, $q(x)$ is continuous, subject to the conditions

$$y(a) = 0, \quad y(b) = 0 \tag{2}$$

for any real or complex λ, always has the trivial solution $y \equiv 0$.

The set of equations (1) and the boundary conditions (2) are called the *Sturm-Liouville boundary-value problem* (1)-(2).

The values of the parameter λ for which the boundary-value problem (1)-(2) has nontrivial solutions $y(x) \not\equiv 0$ are called *eigenvalues*, while the solutions themselves are termed *eigenfunctions of the given boundary-value problem.*

Equation (1) is Euler's equation corresponding to the following conditional-extremum problem: find the minimum of the functional

$$J[y] = \int_a^b (py'^2 + qy^2)\,dx \tag{3}$$

subject to the conditions (2) and the condition

$$\int_a^b y^2(x)\,dx = 1 \tag{4}$$

If some function $y(x)$ is a solution of this variational problem, then it will also be a solution of the problem (1)-(2). This solution will be different from identical zero

by virtue of the condition (4). For this reason, the eigenvalues and eigenfunctions of the Sturm-Liouville boundary-value problem are also called the eigenvalues and eigenfunctions of the functional (3) subject to the conditions (2) and (4).

The eigenfunction $y(x)$ is said to be *normalized* if

$$\int_a^b y^2(x)\,dx = 1$$

Example 1. Find the eigenvalues and the eigenfunctions of the functional

$$J[y] = \int_0^3 [(2x+3)^2 y'^2 - y^2]\,dx \tag{5}$$

subject to the conditions

$$y(0) = 0, \quad y(3) = 0, \tag{6}$$

$$\int_0^3 y^2(x)\,dx = 1 \tag{7}$$

Solution. The Sturm-Liouville equation is of the form

$$-y - \frac{d}{dx}[(2x+3)^2 y'] = \lambda y$$

or

$$(2x+3)^2 y'' + 4(2x+3) y' + (\lambda+1) y = 0 \tag{8}$$

By means of the substitution $2x + 3 = e^t$, equation (8) is reduced to a linear equation ([14], p. 143) with constant coefficients:

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + (\lambda+1) y = 0 \tag{9}$$

Its characteristic equation

$$4k^2 + 4k + \lambda + 1 = 0 \tag{10}$$

has the roots

$$k_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-\lambda}$$

We consider three cases.

(1) $\lambda < 0$. The general solution of equation (9) is then

$$y(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

where k_1 and k_2 are real numbers, and, hence, the general solution of equation (8) is

$$y(x) = C_1 (2x+3)^{k_1} + C_2 (2x+3)^{k_2}$$

The boundary conditions (6) yield

$$\left.\begin{aligned} C_1 3^{k_1} + C_2 3^{k_2} &= 0, \\ C_1 9^{k_1} + C_2 9^{k_2} &= 0 \end{aligned}\right\}$$

whence $C_1 = 0$, $C_2 = 0$, and $y(x) \equiv 0$.

(2) $\lambda = 0$. Then

$$y(t) = (C_1 + C_2 t)\, e^{-\frac{t}{2}}$$

and, hence,

$$y(x) = [C_1 + C_2 \ln(2x+3)] \frac{1}{\sqrt{2x+3}}$$

From the boundary conditions we obtain

$$\left.\begin{aligned} C_1 + C_2 \ln 3 &= 0, \\ C_1 + C_2 \ln 9 &= 0 \end{aligned}\right\}$$

whence $C_1 = 0$, $C_2 = 0$ and, hence, $y(x) \equiv 0$.

(3) $\lambda > 0$. Then $k_{1,2} = -\frac{1}{2} \pm i \frac{\sqrt{\lambda}}{2}$ and the general solution of equation (9) is

$$y(t) = e^{-\frac{t}{2}} \left(C_1 \cos \frac{\sqrt{\lambda}}{2} t + C_2 \sin \frac{\sqrt{\lambda}}{2} t \right)$$

Passing to the variable x, we obtain

$$y(x) = \frac{C_1 \cos\left[\frac{\sqrt{\lambda}}{2} \ln(2x+3)\right] + C_2 \sin\left[\frac{\sqrt{\lambda}}{2} \ln(2x+3)\right]}{\sqrt{2x+3}} \tag{11}$$

The boundary conditions (6) yield

$$\left.\begin{aligned} C_1 \cos\left(\frac{\sqrt{\lambda}}{2} \ln 3\right) + C_2 \sin\left(\frac{\sqrt{\lambda}}{2} \ln 3\right) &= 0, \\ C_1 \cos\left(\frac{\sqrt{\lambda}}{2} \ln 9\right) + C_2 \sin\left(\frac{\sqrt{\lambda}}{2} \ln 9\right) &= 0 \end{aligned}\right\} \tag{12}$$

The system (12) will have nontrivial solutions when its determinant is zero:

$$\begin{vmatrix} \cos\left(\frac{\sqrt{\lambda}}{2}\ln 3\right) & \sin\left(\frac{\sqrt{\lambda}}{2}\ln 3\right) \\ \cos\left(\frac{\sqrt{\lambda}}{2}\ln 9\right) & \sin\left(\frac{\sqrt{\lambda}}{2}\ln 9\right) \end{vmatrix} = 0$$

or $\sin\left(\sqrt{\lambda}\ln 3 - \frac{\sqrt{\lambda}}{2}\ln 3\right) = 0$, that is, $\sin\left(\frac{\sqrt{\lambda}}{2}\ln 3\right) = 0$, whence $\frac{\sqrt{\lambda}}{2}\ln 3 = n\pi$. The eigenvalues are

$$\lambda_n = \frac{4n^2\pi^2}{\ln^2 3} \quad (n = 1, 2, \ldots)$$

Taking any equation of system (12), say the first, and substituting λ_n into it in place of λ, we get

$$C_1 \cos n\pi + C_2 \sin n\pi = 0$$

or $C_1(-1)^n = 0$, whence $C_1 = 0$. Putting $C_1 = 0$ and $\lambda_n = \frac{4n^2\pi^2}{\ln^2 3}$ into (11), we get the eigenfunctions of the given problem:

$$y_n(x) = C_n \frac{\sin\left[\frac{n\pi \ln(2x+3)}{\ln 3}\right]}{\sqrt{2x+3}} \quad (n = 1, 2, \ldots)$$

We find the coefficients C_n from the normalization condition

$$\int_0^3 y_n^2(x)\,dx = 1$$

which yields

$$C_n = \pm\frac{2}{\sqrt{\ln 3}}$$

and, hence,

$$y_n(x) = \pm\frac{2}{\sqrt{\ln 3}} \frac{\sin\left[\frac{n\pi \ln(2x+3)}{\ln 3}\right]}{\sqrt{2x+3}} \quad (n = 1, 2, \ldots)$$

Find the eigenvalues and the eigenfunctions in the following problems.

224. $J[y]=\int_0^1 (y^2+y'^2)\,dx; \quad y(0)=y(1)=0, \quad \int_0^1 y^2\,dx=1.$

225. $J[y]=\int_1^2 x^2y'^2\,dx; \quad y(1)=y(2)=0, \quad \int_1^2 y^2\,dx=1.$

226. $J[y]=\int_1^e (6y^2+x^2y'^2)\,dx; \quad y(1)=y(e)=0, \quad \int_1^e y^2\,dx=1.$

227. $J[y]=\int_\pi^{2\pi} (y^2-y'^2)\,dx; \quad y(\pi)=y(2\pi)=0, \quad \int_\pi^{2\pi} y^2\,dx=1.$

228. $J[y]=\int_0^1 [3y^2-(x+1)^2y'^2]\,dx; \quad y(0)=y(1)=0, \quad \int_0^1 y^2\,dx=1.$

The eigenvalues and eigenfunctions of the variational problem (3), (2), (4) have a number of important properties.

(1) If λ_m and λ_n are two distinct eigenvalues of the functional (3) subject to the conditions (2) and (4), and $y_m(x)$ and $y_n(x)$ are the corresponding eigenfunctions, then these functions $y_m(x)$ and $y_n(x)$ are orthogonal, that is,

$$\int_a^b y_m(x)\,y_n(x)\,dx=0 \quad (m\neq n)$$

(2) All eigenvalues λ_n of the functional (3) are real.

(3) If λ_n is an eigenvalue of the functional (3) and $y_n(x)$ is the corresponding normalized eigenfunction, then

$$J[y_n(x)]=\lambda_n$$

(4) The smallest of the eigenvalues coincides with the minimum of the functional (3) subject to the conditions (2) and (4).

Example 2. Prove the inequality

$$\int_0^{\pi} y'^2(x)\,dx \geqslant \int_0^{\pi} y^2(x)\,dx,\ y(0) = y(\pi) = 0$$

Solution. We find $\min \int_0^{\pi} y'^2(x)\,dx$ subject to the conditions

$$\int_0^{\pi} y^2(x)\,dx = 1, \quad y(0) = y(\pi) = 0$$

Euler's equation for the functional

$$J[y] = \int_0^{\pi} [y'^2(x) - \lambda y^2(x)]\,dx$$

is of the form

$$y'' + \lambda y = 0;\ y(0) = 0,\ y(\pi) = 0$$

The eigenfunctions of this problem are $y_n(x) = \sin nx$, and the eigenvalues are $\lambda_n = n^2$.

The smallest eigenvalue is $\lambda_1 = 1$. Therefore, by property (4),

$$\min \int_0^{\pi} y'^2(x)\,dx = 1$$

Hence, for any function $y(x)$ for which $\int_0^{\pi} y^2(x)\,dx = 1$ we have

$$\int_0^{\pi} y'^2(x)\,dx \geqslant \int_0^{\pi} y^2(x)\,dx$$

This inequality cannot be improved since for $y_1(x) = \frac{\sin x}{\sqrt{\pi}}$ we have

$$\int_0^{\pi} y_1'^2(x)\,dx = \int_0^{\pi} y_1^{2'}(x)\,dx = 1$$

Note. If $\int_0^{\pi} y^2(x)\,dx = k^2 \neq 1$, then the problem reduces to the preceding one by introduction of the function $z(x) = \frac{y(x)}{k}$.

Using the extremal definition of eigenvalues, we will indicate ways of their approximate computation via the Ritz method. Bear in mind that the Ritz method yields for the eigenvalue an approximation with excess.

Example 3. Approximate the first eigenvalue of the problem

$$y'' + \lambda^2 y = 0,$$
$$y(-1) = y(1) = 0$$

Solution. The problem of minimizing the functional

$$\int_{-1}^{1} y'^2\,dx$$

subject to the conditions

$$y(-1) = y(1) = 0 \quad \text{and} \quad \int_{-1}^{1} y^2(x)\,dx = 1$$

is an isoperimetric problem and reduces to the problem of minimizing the functional

$$J = \int_{-1}^{1} (y'^2 - \lambda^2 y^2)\,dx$$

for which Euler's equation coincides with the given differential equation $y'' + \lambda^2 y = 0$, $y(-1) = y(1) = 0$.

The general solution of the equation is $y(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$. From the boundary conditions we find

$$\left.\begin{aligned} C_1 \cos \lambda - C_2 \sin \lambda = 0, \\ C_1 \cos \lambda + C_2 \sin \lambda = 0 \end{aligned}\right\} \tag{13}$$

so that the condition for the existence of a nontrivial solution of the system (13) is the condition $\sin 2\lambda = 0$ or $\lambda = \frac{n\pi}{2}$.

Thus, for the first eigenvalue we have $\lambda_1^2 = \left(\frac{\pi}{2}\right)^2$ and the fundamental tone of a string is exactly given by the solution $y = \cos\frac{\pi x}{2}$, $\lambda = \frac{\pi}{2}$; the first overtone is $y = \sin \pi x$, $\lambda = \pi$; the second overtone is $y = \cos\frac{3\pi x}{2}$, $\lambda = \frac{3}{2}\pi$, and so forth.

For the sake of comparison, we will seek approximate even solutions (even tones of the string) in the form of a polynomial in even powers of x. Let us take the coordinate functions in the form $\varphi_k(x) = x^{2k-2} - x^{2k}$ $(k = 1, 2, \ldots)$ and minimize the functional J on the functions $y_m(x) = \sum_{k=1}^{m} c_k \varphi_k(x)$. Confining ourselves to $y_1(x) = c_1\varphi_1(x)$, we will have $J = c_1^2\left(\frac{8}{3} - \frac{16}{15}\lambda^2\right)$ so that for a determination of c_1 we get

$$\frac{\partial J}{\partial c_1} = 2c_1\left(\frac{8}{3} - \frac{16}{15}\lambda^2\right) = 0$$

And since it must be true that $c_1 \neq 0$, it follows that $\lambda^2 = 2.5$. Taking for y

$$y = c_1\varphi_1(x) + c_2\varphi_2(x)$$

we find

$$J = c_1^2\left(\frac{8}{3} - \frac{16}{15}\lambda^2\right) + 2c_1c_2\left(\frac{8}{15} - \frac{16}{105}\lambda^2\right) + c_2^2\left(\frac{88}{105} - \frac{16}{315}\lambda^2\right)$$

and for a determination of c_1 and c_2 we obtain the system

$$\left.\begin{aligned} \frac{\partial J}{\partial c_1} &= c_1\left(\frac{16}{3} - \frac{32}{15}\lambda^2\right) + c_2\left(\frac{16}{15} - \frac{32}{105}\lambda^2\right) = 0, \\ \frac{\partial J}{\partial c_2} &= c_1\left(\frac{16}{15} - \frac{32}{105}\lambda^2\right) + c_2\left(\frac{176}{105} - \frac{32}{315}\lambda^2\right) = 0 \end{aligned}\right\}$$

The condition for the existence of nontrivial solutions c_1 and c_2 of this system is that the determinant of the system be equal to zero, which yields $\lambda^4 - 28\lambda^2 + 63 = 0$, whence $\lambda_1^2 = 2.46744$, $\lambda_2^2 = 25.53256$. Compare the approximate values of λ_1^2 and λ_2^2 thus obtained with the exact values. The exact value of λ_1^2 is $\left(\frac{\pi}{2}\right)^2 \approx 2.46740$ and that of λ_2^2

is $\left(\frac{3\pi}{2}\right)^2 \approx 22.20661$ so that the approximation of λ_1^2 is extremely exact, whereas the approximation is rough with respect to the second eigenvalue.

Example 4. Find the first eigenvalue of the problem

$$y'' + \lambda (1 + x^2) y = 0, \quad y(-1) = y(1) = 0$$

Solution. For the coordinate functions, take the functions $\varphi_k(x) = 1 - x^{2k}$ $(k = 1, 2, \ldots)$, which clearly satisfy the boundary conditions. Assuming

$$y(x) = c_1 (1 - x^2) + c_2 (1 - x^4)$$

we pose the problem of minimizing the functional

$$J[y] = \int_{-1}^{1} [y'^2 - \lambda (1 + x^2) y^2] \, dx$$

for which the given equation is Euler's equation. We have

$$J = c_1^2 \left(\frac{8}{3} - \frac{128}{105}\lambda\right) + 2c_1c_2 \left(\frac{16}{5} - \frac{64}{45}\lambda\right) + c_2^2 \left(\frac{32}{7} - \frac{5888}{3465}\lambda\right)$$

To determine c_1 and c_2 we get the system

$$\left.\begin{aligned} \frac{\partial J}{\partial c_1} &= 2c_1 \left(\frac{8}{3} - \frac{128}{105}\lambda\right) + 2c_2 \left(\frac{16}{5} - \frac{64}{45}\lambda\right) = 0, \\ \frac{\partial J}{\partial c_2} &= 2c_1 \left(\frac{16}{5} - \frac{64}{45}\lambda\right) + 2c_2 \left(\frac{32}{7} - \frac{5888}{3465}\lambda\right) = 0 \end{aligned}\right\}$$

The condition for the existence of a nontrivial solution of this system yields

$$52\lambda^2 - 1068\lambda + 2079 = 0$$

whence, taking the smaller root, we find $\lambda_1 = 2.1775$.

Rayleigh's principle. Suppose we have an eigenvalue problem

$$L(y) \equiv -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x) y = \lambda r(x) y, \qquad (14)$$

$$\left.\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= 0, \quad \alpha_1^2 + \beta_1^2 > 0, \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0, \quad \alpha_2^2 + \beta_2^2 > 0 \end{aligned}\right\} \qquad (15)$$

where $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous on $[a, b]$; $p(x) > 0$ on [a, b].

We call the function $y(x)$ *admissible* ($y \in D$) if it is twice continuously differentiable and satisfies the boundary conditions (15).

Suppose that for every admissible function $y(x)$ the following condition is fulfilled:

$$\int_a^b yL(y)\,dx \geqslant 0$$

Then the boundary-value problem (14)-(15) has only real-valued eigenvalues λ.

To the eigenvalue problem we can associate the following variational problem.

From among all admissible functions $y(x)$ such that

$$\int_a^b r(x)\,y^2\,dx > 0 \tag{16}$$

find the function for which $\dfrac{\int_a^b yL(y)\,dx}{\int_a^b r(x)\,y^2\,dx} = \min.$

Let $y = \psi_1(x)$ be a solution of this problem.

If λ_1 is a minimal value, that is, if

$$\lambda_1 = \min_{y \in D} \frac{\int_a^b yL(y)\,dx}{\int_a^b r(x)\,y^2\,dx} = \frac{\int_a^b \psi_1 L(\psi_1)\,dx}{\int_a^b r\psi_1^2\,dx}$$

then λ_1 is the least positive eigenvalue and $\psi_1(x)$ is the corresponding eigenfunction.

If, in addition to the condition (16), we impose another condition on the admissible functions,

$$\int_a^b r\psi_1 y\,dx = 0$$

(orthogonality condition), then the problem

$$\frac{\int_a^b yL(y)\,dx}{\int_a^b ry^2\,dx} = \min$$

will again have a certain solution $\psi_2(x)$.

If λ_2 is the corresponding minimal value, then λ_2 will be the eigenvalue next in magnitude ($\lambda_2 \geqslant \lambda_1$) and $\psi_2(x)$ will be the corresponding eigenfunction orthogonal to $\psi_1(x)$. Generally, if we already know the first k positive eigenvalues

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k$$

and the corresponding orthogonal system of eigenfunctions

$$\psi_1(x),\ \psi_2(x),\ \ldots,\ \psi_k(x)$$

then the following eigenvalue is equal to

$$\lambda_{k+1} = \min_{y \in D} \frac{\int_a^b yL(y)\,dx}{\int_a^b ry^2\,dx}$$

Now we consider the admissible functions $y(x)$ for which, besides (16), the following accessory conditions are fulfilled:

$$\int_a^b r(x)\,\psi_\nu(x)\,y(x)\,dx = 0 \quad (\nu = 1,\ 2,\ \ldots,\ k)$$

If in the equation (14) the function $r(x) > 0$ on $[a,\ b]$, then for an upper estimate of the smallest positive eigenvalue λ_1 one often makes use of the following inequality (Rayleigh's principle):

$$\lambda_1 \leqslant \frac{\int_a^b yL(y)\,dx}{\int_a^b ry^2\,dx}$$

Example 5. Use the Rayleigh principle to estimate λ_1 for the following boundary-value problem:

$$-y'' = \lambda y; \quad y'(0) = 0, \quad y(1) = 0$$

Solution. In our case, $L(y) = -y''$, that is, $p(x) \equiv 1 > 0$, $q(x) \equiv 0$ and $r(x) \equiv 1 > 0$ on $[0, 1]$. Clearly, $\alpha_1 = 0$, $\beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = 0$ so that $\alpha_1^2 + \beta_1^2 = 1 > 0$, $\alpha_2^2 + \beta_2^2 = 1 > 0$. We take $y(x) = 1 - x^2$ for an admissible function. By Rayleigh's principle we have

$$\lambda_1 \leqslant \frac{\int_0^1 yL(y)\,dx}{\int_0^1 ry^2\,dx} = \frac{\int_0^1 2(1-x^2)\,dx}{\int_0^1 (1-x^2)\,dx} = \frac{\frac{4}{3}}{\frac{8}{15}} = 2.5$$

Note that the exact value is $\lambda_1 = \frac{\pi^2}{4} \approx 2.4674$.

In the following problems, estimate the smallest eigenvalues.

229. $-y'' = \lambda(10 - x^2)y$; $y(-1) = y(1) = 0$.

230. $-y'' = \lambda y$; $y(0) = y(1) = 0$.

The Kantorovich method (the method of reducing to ordinary differential equations) can also be used in the problem of finding eigenvalues and eigenfunctions. For example, suppose we have the equation

$$\Delta z + \lambda z = 0$$

in a domain D and let

$$z|_\Gamma = 0$$

where Γ is the boundary of D.

We seek the solution in the form

$$z_m = \sum_{k=1}^{m} \alpha_k(x)\,\varphi_k(x, y) + \varphi_0(x, y)$$

We choose the coordinate functions $\varphi_k(x, y)$ and the as yet unknown functions $\alpha_k(x)$ so that $z_m(x, y)$ vanish everywhere on Γ. The functions $\alpha_1(x)$, $\alpha_2(x)$, . . ., $\alpha_m(x)$ must satisfy the system of equations

$$\int_{D_x} [\Delta z_m + \lambda z_m]\,\varphi_k(x, y)\,dy = 0 \quad (k = 1, 2, \ldots, m) \quad (17)$$

and vanish for the extreme values of the argument. Here, D_x is a section of the domain D cut by the straight line $x = \text{constant}$.

Those values of D_x for which the system (17) has a nontrivial solution yield approximate eigenvalues, while the solutions themselves yield an approximation to the eigenfunctions.

Example 6. Approximate the first eigenvalue and the first eigenfunction of the problem

$$\Delta z + \lambda z = 0, \quad z|_\Gamma = 0$$

where the domain D is a rectangle: $-a \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$.

Solution. We seek the solution of the problem in the form

$$z_1(x, y) = (y^2 - b^2)\,\alpha_1(x)$$

In this case equation (17) has the form

$$\int_{-b}^{b} [2\alpha_1 + (y^2 - b^2)\,\alpha_1'' + \lambda (y^2 - b^2)\,\alpha_1]\,(y^2 - b^2)\,dy = 0$$

or

$$\frac{16}{15} b^5 \alpha_1'' + \left(\frac{16}{15} b^5 \lambda - \frac{8}{3} b^3\right) \alpha_1 = 0, \tag{18}$$

$$\alpha_1(-a) = \alpha_1(a) = 0$$

The general solution of (18) is

$$\alpha(x) = C_1 \sin \sqrt{\lambda - \frac{5}{2b^2}} \cdot x + C_2 \cos \sqrt{\lambda - \frac{5}{2b^2}} \cdot x$$

Taking into account the symmetry of the problem and choosing a particular solution, we obtain

$$C_1 = 0, \quad C_2 \cos \sqrt{\lambda - \frac{5}{2b^2}}\, a = 0$$

whence it is clear that a nontrivial solution is only obtained if

$$\sqrt{\lambda - \frac{5}{2b^2}}\, a = (2k-1)\,\frac{\pi}{2},$$

$$\lambda = \frac{(2k-1)^2 \pi^2}{(2a)^2} + \frac{5}{2b^2}$$

In particular, for $k=1$ we find

$$\lambda = \frac{\pi^2}{(2a)^2} + \frac{10}{(2b)^2}$$

in place of the exact solution

$$\lambda = \frac{\pi^2}{(2a)^2} + \frac{\pi^2}{(2b)^2}$$

The error is less than 1.3%.

For the first eigenfunction we obtain the approximation

$$z_1(x, y) = (y^2 - b^2) \cos \frac{\pi x}{2a}$$

Approximate the first eigenvalue in the following problems.

231. $y'' + \lambda^2 y = 0; \quad y(0) = y(1) = 0.$

232. $y'' + \lambda (2 + \cos x) y = 0; \quad y(0) = y(\pi) = 0.$

233. Approximate the first eigenvalue of the problem

$$\Delta z + \lambda z = 0, \quad z|_\Gamma = 0$$

where the domain D is a circle of unit radius with centre at the coordinate origin.

ANSWERS AND HINTS

1. (a) $f_{\min}=0$ at point $(0, 0)$; (b) $f_{\max}=1$ at point $(0, 0)$; (c) no extremum. **2.** No extremum. **3.** $f_{\min}=-8$ at the points $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. No extremum at point $(0, 0)$. **4.** $f_{\min}=0$ at point $(0, 0)$. There is a nonstrict maximum at points of the circle $x^2+y^2=1$.

5. $f_{\max}=\sqrt{3}$ at point $(1, -1)$. **6.** $f_{\min}=4$ at point $\left(\frac{1}{2}, 1, 1\right)$. **7.** $f_{\min}=-1$ at point $(1, 0)$. **8.** $f_{\min}=-\frac{3\sqrt{3}}{8}$ at point $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$; $f_{\max}=\frac{3\sqrt{3}}{8}$ at point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

9. $f_{\max}=\left(\frac{1}{n^2+n+2}\right)^{\frac{n^2+n+2}{2}}$ for $x_1=x_2=\ldots=x_n=$ $=\frac{2}{n^2+n+2}$. **11.** No. **13.** The numbers α_k and β_k must be Fourier coefficients of the function $f(x)$.

14. $f_{\min}=-\frac{1}{2}$ at the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; $f_{\max}=\frac{1}{2}$ at the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

15. $f_{\min}=\frac{36}{13}$ at point $\left(\frac{18}{13}, \frac{12}{13}\right)$. **16.** $f_{\min}=4$ at the points $(2, 2, 1)$, $(1, 2, 2)$ and $(2, 1, 2)$; $f_{\max}=4\frac{4}{27}$ at the points $\left(\frac{4}{3}, \frac{4}{3}, \frac{7}{3}\right)$, $\left(\frac{7}{3}, \frac{4}{3}, \frac{4}{3}\right)$ and $\left(\frac{4}{3}, \frac{7}{3}, \frac{4}{3}\right)$.

17. $f_{\max} = e^{\frac{a^2}{4}}$. **18.** $f_{\min} = 1$ at point $\left(\frac{4}{5}, \frac{3}{5}\right)$; $f_{\max} = 11$ at point $\left(-\frac{4}{5}, -\frac{3}{5}\right)$. **19.** $f_{\min} = -9$ at point $(-1, 2, -2)$; $f_{\max} = 9$ at point $(1, -2, 2)$. **20.** $f_{\max} = \frac{1}{8}$ at point $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right)$. **21.** *Hint.* Seek the minimum of the function $z = \frac{1}{2}(x^n + y^n)$ subject to the condition $x + y = S$.

22. c^4. **23.** $\frac{4\sqrt{5}}{5}$. **24.** $\frac{19\sqrt{2}}{8}$. **25.** A square of side $a = R\sqrt{2}$. **26.** The radius of the base of the cylinder is $r = \frac{R}{2}\sqrt{2 + \frac{2}{\sqrt{5}}}$, the altitude is $h = R\sqrt{2 - \frac{2}{\sqrt{5}}}$.

27. First. **28.** Proximity of any order. **29.** Proximity of any order. **30.** $\rho = e^{-1}$.

31. $\rho = 1$. **32.** $\rho = e - 1$. **33.** $\rho_1 = e - 1$. **34.** $\rho_2 = \frac{2\pi + 3}{6}$.

35. $\rho_{1001} = e$. **36.** Continuous. **37.** Continuous. **38.** Discontinuous (consider the sequence $y_n(x) = \frac{\sin nx}{n}$).

39. (a) Discontinuous; (b) continuous. **40.** (a) Discontinuous; (b) continuous.

41. Continuous. **45.** $\Delta J = \frac{1 - e^2}{2}$.

48.

α	ΔJ	δJ	
			$\Delta J = \alpha + \frac{\alpha^2}{5}$;
1	1.2	1	
−0.1	−0.098	−0.1	$\delta J = \alpha$.
0.01	0.01002	0.01	

49. $\Delta J = \frac{3(e^2-1)}{4}\alpha + 6(3-e)\alpha^2 + \frac{\alpha^3}{5}$; $\delta J = \frac{3(e^2-1)}{4}\alpha$

α	δJ	ΔJ
1	4.7919	6.6821
0.1	0.4792	0.4963
0.01	0.0479	0.0481

50. (1) Yes; (2) yes; (3) yes; (4) no.

51. $\delta J^2[y] = 2J[y] \cdot \delta J[y]$.

53. $\Delta J = 3k + \frac{e}{e-1} k^2$; $\delta J = 3k$

k	ΔJ	δJ
1	4.582	3
0.1	0.3158	0.3
0.01	0.03016	0.03

54. $\Delta J = \frac{5}{3} k + \frac{8}{7} k^2$; $\delta J = \frac{5}{3} k$

k	ΔJ	δJ
1	2.810	1.667
0.1	0.181	0.167
0.01	0.0168	0.0167

55. $\Delta J = \frac{4}{3} k^2$; $\delta J \equiv 0$

k	δJ	ΔJ
-1	0	1.3333
0.3	0	0.1200
0.03	0	0.0012

57. $\delta J = \int_a^b \delta y \, dx$. **58.** $\delta J = 2 \int_a^b (y \, \delta y - y' \, \delta y') \, dx$.

59. $\delta J = 2y(0) \cdot \delta y(0) + \int_0^1 (x \, \delta y + 2y' \delta y') \, dx$.

60. $\delta J = \int_0^{\pi} (y' \cos y \, \delta y + \sin y \, \delta y') \, dx$.

61. $\delta J = \int_a^b \left(\frac{\partial f}{\partial y_1} \delta y_1 + \frac{\partial f}{\partial y_2} \delta y_2 + \ldots + \frac{\partial f}{\partial y_n'} \delta y_n' \right) dx$.

62. $\delta^2 J[y, y] = 2J[\delta y, \delta y]$.

63. $\delta^2 e^{F(y)} = e^{F(y)}\left((\delta F)^2 + \delta^2 F\right)$.

65. $\delta^2 J = \int\limits_a^b \sum\limits_{k,\ l=0}^{m} \frac{\partial^2 F}{\partial y^{(k)}\,\partial y^{(l)}}\,\delta y^{(k)}\delta y^{(l)}\,dx$.

66. $\delta^2 J = \iint\limits_G \left[F''_{zz}(\delta z)^2 + F''_{zz'_x}\delta z\delta z'_x + \ldots + F''_{z'_y z'_y}(\delta z'_y)^2\right] dx\,dy$.

67. $\delta^2 J = \int\limits_a^b \Big[\sum\limits_{i,\ k=1}^{n} F''_{y_i y_k}\delta y_i\delta y_k + \sum\limits_{i,\ k=1}^{n} F''_{y_i y'_k}\delta y_i\delta y'_k + \sum\limits_{i,\ k=1}^{n} F''_{y'_i y'_k}\delta y'_i\delta y'_k\Big]\,dx$.

68. Introduce the functional

$$J[\varphi + \alpha\eta] = \Phi(\alpha)$$

and take advantage of the second definition of a variation. The requirement $\delta J = 0$ leads to the integral equation

$$\int\limits_a^b K(s, t)\,\varphi(s)\,ds + \varphi(t) - f(t) = 0$$

69. Proceeding as in the preceding example, we find that Euler's functional equation, which expresses the fact that the first variation vanishes, has the following form:

$$(p\varphi')' - \varphi(x+2) - \varphi(x-2) + \varphi(x) + f(x) = 0$$

This is a mixed differential-difference equation.

70. $-(p\varphi')' + q\varphi = f(x)$. **71.** $y = -x^3$.

72. $y = \frac{\sinh(2-x)}{\sinh 1}$.

73. Two extremals

$$y = \frac{1 + (3 \pm 2\sqrt{2})(2x-1)^2}{4(\sqrt{2} \pm 1)}$$

74. Two extremals $y = \sqrt[3]{(x+1)^2}$, $y = \sqrt[3]{(3x-1)^2}$.

75. $y = (C + x)\sin x$, where C is an arbitrary constant.

76. $y=\frac{1}{2}[e^{-x}+(1+e)\,xe^{-x}-1]$.

77. $y=\frac{7}{6}x-\frac{1}{6}x^3$.

78. $y=\frac{13}{6}x-\frac{1}{6}x^3+2$. **79.** $y=\ln x$. **81.** The integral does not depend on the path of integration; the variational problem is meaningless. **82.** $y=0$ if $\alpha=0$; for $\alpha\neq 0$ there is no smooth extremal. **83.** $y=\cos x$. **84.** $y=\cos x+C\sin x$, where C is an arbitrary constant. **85.** $y=x+1$.

86. $y=\frac{\sinh x}{\sinh 1}$. **87.** $y=e^{2(1-x)}$.

88. There are no extremals; Euler's equation has no solutions. **89.** $y=C_1+C_2x-\frac{x^2}{4}$. **90.** No extremals.

93. $y=C_1e^x+C_2e^{-x}+\frac{1}{2}xe^x$. **94.** $y=2\cosh x$.

96. $y=\frac{y_1\sin x}{\sin x_1}$. **97.** $y=2x$.

98. The circle $\frac{1}{r}=K$. **99.** $y=(1-x)\sinh x$.

100. $y=\frac{x^3}{6}(x^3+6x+1)$.

101. There is no extremum. **102.** The variational problem is meaningless since there is a total differential under the integral sign: **103.** $y(x)=\sinh x$.

104. $y=\frac{1}{2}x^2$. **105.** $\begin{cases} y(x)=\sin 2x, \\ z(x)=-\frac{x^2}{2}+\frac{32+\pi^2}{8\pi}x. \end{cases}$

106. $\begin{cases} y(x)=-\frac{1}{6}(x^3+5x-6), \\ z(x)=x. \end{cases}$ **107.** $\begin{cases} y(x)=\sin x, \\ z(x)=\sin x. \end{cases}$

108. $\begin{cases} y(x)=\frac{x^2}{2}+1, \\ z(x)=1. \end{cases}$

110. $\left(\frac{\partial z}{\partial x}\right)^2\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial y}\right)^2\frac{\partial^2 z}{\partial y^2}=f(x, y)$.

111. $\Delta\Delta r = 0$.

112. $\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(a_j(x_1, \ldots, x_n)\frac{\partial z}{\partial x_j}\right) + c(x_1, \ldots, x_n)\, z = f(x_1, \ldots, x_n)$.

113. *Solution.* The problem is posed as follows; from among the surfaces $z = f(x, y)$ located above the domain D of the xy-plane and passing through a given closed space curve having for its projection the boundary curve Γ of D, find the surface whose area

$$S = \iint_D \sqrt{1 + \varphi_x^2 + \varphi_y^2}\, dx\, dy$$

is a minimum. For this problem, Euler's differential equation is

$$\frac{\partial}{\partial x}\frac{\varphi_x}{\sqrt{1+\varphi_x^2+\varphi_y^2}} + \frac{\partial}{\partial x}\frac{\varphi_y}{\sqrt{1+\varphi_x^2+\varphi_y^2}} = 0$$

or, expanded,

$$\varphi_{xx}(1 + \varphi_y^2) - 2\varphi_{xy}\varphi_x\varphi_y + \varphi_{yy}(1 + \varphi_x^2) = 0$$

This is the desired differential equation of minimal surfaces. The physical realization of a minimal surface is given, for example, by a soap film spanned over a wire loop.

114. $z(x, y) = y$. The problem has a unique solution although the boundary conditions are not specified over the entire boundary.

115. $r\cos\varphi + C_2 = C_1 \ln\left|r\sin\varphi + \sqrt{r^2\sin^2\varphi - C_1^2}\right|$.

117. $x^2\cos C_2 - y^2\cos C_2 - 2xy\sin C_2 = C_1$.

118. Central field. **119.** (a) Proper field, (b) central field, (c) a field is not formed. **120.** Proper field. **121.** (a) Central field, (b) a field is not formed, (c) proper field. **122.** (a) Central field, (b) proper field, (c) a field is not formed. **123.** A field is not formed since this family of curves does not cover the entire domain D. **124.** $y = C_1\cosh x$ form a proper field of extremals; $y = C_2\sinh x$ form a central field of extremals.

125. $y = C\cos x$ form a proper field of extremals; $y = C\sin x$ form a central field of extremals. **126.** The extremal $y = \frac{x}{6}(1 - x^2)$ is included in the central field of

extremals $y = C_1 x - \frac{x^3}{6}$ with centre at the point O (0, 0).

127. The extremal $y = e^x$ may be included in the proper field of extremals $y = e^x + C$. **128.** If $a < \pi$, then the extremal $y = 0$ may be included in the central field of extremals $y = C \sin x$ with centre at the point O (0, 0). For $a > \pi$ the family of curves $y = C \sin x$ does not form a field. **129.** The extremal $y = x + 1$ is included in the proper field $y = x + C$.

130. $y = -\frac{x^2}{4}$. **131.** $y\left(\frac{y}{4} - x\right) = 0$. **132.** $y^2 - 1 = 0$.

133. O^* (1, 0). **134.** There is no conjugate point. **135.** The Jacobi condition is fulfilled. **136.** The Jacobi condition is fulfilled for any a. **137.** The Jacobi condition is fulfilled. The extremal $y = 0$ may be included both in the central field and in the proper field of extremals. **138.** The Jacobi condition is fulfilled. The extremal $y = \frac{b-1}{a} x + 1$ may be included in the central field of etremals with centre at the point A (0, 1). **139.** The Jacobi condition is not fulfilled. **142.** Yes. **143.** Yes. **144.** Yes. **145.** Yes, but Legendre's condition is only fulfilled for $\frac{b}{a} < 1$. **146.** A strong minimum is attained on the function $y = e^x$. **147.** A strong minimum is attained on the function $y = 2 \ln (x + 1)$. **148.** A weak minimum is attained on the function $y = x^2$. **149.** A weak minimum is attained on the straight line $y = \frac{b}{a} x$. **150.** A strong minimum is attained on the curve $y = \frac{\ln (1+x)}{\ln 2}$. **151.** A strong maximum is attained on the curve $y = \cos x + \sin x$. **152.** An extremum is not attained on continuous curves. **153.** A weak minimum is attained on the straight line $y = 2x + 1$. There is no strong extremum. **154.** A strong minimum is attained on the extremal $y = 2x - 1$. **155.** A strong minimum is attained on the extremal $y = x^2$. **156.** A weak minimum is attained on the extremal $y = x - 1$. **157.** A weak minimum is attained on the extremal $y = \frac{b}{a} x$ for $|b| < \frac{a}{\sqrt{2}}$ and a weak maximum for $|b| > \frac{a}{\sqrt{2}}$. For $|b| = \frac{a}{\sqrt{2}}$, an extremum is not attained.

158. A weak minimum is attained on the extremal $y = \sqrt[3]{[x(q^{3/2} - p^{3/2}) + p^{3/2}]^2}$ for $p \neq q$; for $p = q$, the extremal that gives the functional a weak minimum is the straight line $y = p$. **159.** (a) For $\varepsilon > 0$, the extremal $y = \dfrac{\sinh \dfrac{x}{\sqrt{\varepsilon}}}{\sinh \dfrac{1}{\sqrt{\varepsilon}}}$ gives the functional a strong minimum.

(b) For $\varepsilon < 0$, $|\varepsilon| > \dfrac{1}{\pi^2}$, the extremal $y = \dfrac{\sin \dfrac{x}{\sqrt{|\varepsilon|}}}{\sin \dfrac{1}{\sqrt{|\varepsilon|}}}$ gives the functional a strong maximum. (c) For $\varepsilon = 0$, there is no solution of the extremal problem in the class of continuous functions.

Consider the function $y_\varepsilon(x) = e^{\frac{x-1}{\sqrt{\varepsilon}}}$ $(\varepsilon > 0)$ which is a solution of Euler's equation $\varepsilon y'' - y = 0$ for the given functional. The function $y_\varepsilon(x)$ satisfies the boundary condition $y(1) = 1$ exactly but does not satisfy the second boundary condition $y(0) = 0$. However, $\lim\limits_{\varepsilon \to 0} y_\varepsilon(0) = 0$. When $\varepsilon \to 0$ we obtain from $y_\varepsilon(x)$ the "limiting solution"

$$y(x) = \begin{cases} 0, & 0 \leqslant x < 1, \\ 1, & x = 1 \end{cases}$$

160. The extremal $y = -\dfrac{2 \ln(1+x)}{\ln 2}$ gives a strong minimum.

161. There is a strong minimum on the extremal $y(x) = 1$.

162. A weak minimum is attained on the extremal $y(x) = \dfrac{b}{a} x$ for $\dfrac{b}{a} < \dfrac{\sqrt{3}}{2}$, for $\dfrac{b}{a} > \dfrac{\sqrt{3}}{2}$ a weak maximum is attained, and for $\dfrac{b}{a} = \dfrac{\sqrt{3}}{2}$ even a weak extremum is not attained.

163. A weak minimum is attained on the straight line $y = \dfrac{b}{a} x$ when $b < a$, for $b > a$ a weak maximum is attained, and for $b \geqslant a\sqrt{3}$ a strong maximum is attained. For $b < a\sqrt{3}$ there is neither a strong minimum nor a strong

maximum. **164.** On the extremal $y = 2x$, $z = 4x$ is attained a weak minimum.

165. The extremal is the parabola $\left.\begin{aligned} y &= x, \\ z &= x^2 - x \end{aligned}\right\}$, which is included in the central field of extremals

$$\left.\begin{aligned} y &= \alpha x, \\ z &= x^2 + \beta x \end{aligned}\right\} \qquad (S)$$

(α, β are parameters), with centre at the point (0, 0, 0). That Legendre's strong conditions are fulfilled is obvious. We will show that on the interval $0 \leqslant x \leqslant 1$ there is no point x^* conjugate to the point $x = 0$. For this purpose, it suffices to see that the extremals of the family (S) do not intersect the given extremal for $x \in [0, 1]$. Indeed, suppose that at the point $x^* \in [0, 1]$ any two extremals of the family (S) intersect. Then we will have

$$\left.\begin{aligned} \alpha_1 x^* &= \alpha_2 x^*, \\ x^{*2} + \beta_1 x^* &= x^{*2} + \beta_2 x^* \end{aligned}\right\}$$

From this it follows that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Hence, no two distinct extremals can intersect. Thus, Jacobi's strong condition is fulfilled on the interval [0, 1] and, generally, on any interval of finite length.

166. The family of extremals $y(x) = C_1 \cosh\frac{x - C_2}{C_1} - \lambda$. The arbitrary constants C_1, C_2 and the parameter λ are determined from the conditions

$$y_0 = C_1 \cosh\frac{x_0 - C_2}{C_1} - \lambda, \quad y_1 = C_1 \cosh\frac{x_1 - C_2}{C_1} - \lambda,$$

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx = C_1\left(\sinh\frac{x_1 - C_2}{C_1} - \sinh\frac{x_0 - C_2}{C_1}\right) = l.$$

167. $y(x) = 3x^2 + 2x + 1$. **168.** $y(x) = \pm 2 \sin n\pi x$, where n is an integer. **169.** $y(x) = \frac{1}{4}(2x - x^2)$. **170.** $\sqrt{6}$. **171.** $r = R$, $z = C_1 + C_2\varphi$. **172.** $\frac{4}{\sqrt{5}}$. **173.** $\sqrt{20}$. **174.** $2\sqrt{2} - 1$. **175.** $\frac{\sqrt{10}}{10}$. **178.** $\frac{\sqrt{11}}{2}$. **179.** $\sqrt{17 + 4\sqrt{6}}\left(\frac{5}{2} - \sqrt{6}\right)$.

180. 1. **181.** If $\cos x_1 \neq 0$, then the extremum may be attained only on the straight line $\left.\begin{matrix} y = 0, \\ z = 0 \end{matrix}\right\}$. But if $\cos x_1 = 0$, that is, $x_1 = \frac{\pi}{2} + n\pi$, where n is an integer, then $y = C_4 \sin x$, $z = -C_4 \sin x$, where C_4 is an arbitrary constant. **182.** $J(A, B) = 4 \coth 1$. **183.** $J(A, B) = \frac{26}{5}$.

184. $y = 2x^{\frac{2}{3}}$. **185.** Polygonal lines composed of segments of straight lines $y = x$ and $y = 1$ or of segments of the straight lines $y = 0$ and $y = x - 1$ yield an absolute minimum. The straight line $y = \frac{1}{2} x$ yields a weak maximum.

186. $y = -x$ for $0 \leqslant x \leqslant 1$; $y = x - 2$ for $1 < x \leqslant 4$ and $y = x$ for $0 \leqslant x \leqslant 3$; $y = -x + 6$ for $3 < x \leqslant 4$. The functional attains an absolute minimum on both polygonal lines.

187. No. **188.** $y = \begin{cases} 0, & x \leqslant 0, \\ x, & x > 0. \end{cases}$

189. The extremals are straight lines. If $\left|\frac{y_2 - y_1}{x_2 - x_1}\right|_* < 1$, then there are two discontinuous solutions: polygonal lines parallel to the bisectors of the quadrantal angles. **190.** The straight line $y = x \tan \varphi$ that joins the given points yields a weak maximum if $0 < \tan \varphi < \pi$, a weak minimum if $\pi < \tan \varphi < 2\pi$, and so on. The polygonal line composed of segments of the straight lines, the tangent of the angles of inclination of which is equal to $\frac{4n-1}{2} \pi$ (n an integer) gives a strong minimum.

191.

$$y(x) = \begin{cases} \pm \frac{3}{4} x, & 0 \leqslant x \leqslant \frac{16}{5}, \\ \pm \sqrt{9 - (x-5)^2}, & \frac{16}{5} < x \leqslant \frac{34}{5}, \\ \pm \frac{3(x-10)}{4}, & \frac{34}{5} < x \leqslant 10 \end{cases}$$

192. The extremals are the ellipses

$$\frac{(x + C_1)^2}{C_2^4} + \frac{y^2}{C_2^2} = 1 \qquad \text{(I)}$$

with centres on the x-axis. The boundary of the admissible domain is determined by the equations $y = 0$ and $y^2 = \pm 2(x - C_3)$ (the latter is a solution of the equation $1 - y^2y'^2 = 0$). The parameters C_1 and C_2 are chosen so that the ellipse (I) passes through the given points A and B. The functional attains a maximum on an arc of the ellipse. If the path from A to B is chosen along the arcs of two parabolas (and, possibly, along a segment of the straight line $y = 0$), then we obtain a discontinuous solution on which the functional attains a minimum ($\min J = 0$).

193. $\frac{dy}{dx} = \frac{x^2y^2}{4p^2}$, $\frac{dp}{dx} = \frac{x^2y}{2p}$. **194.** $\frac{dy}{dx} = \frac{p}{2xy}$, $\frac{dp}{dx} = \frac{p^2}{4xy^2}$.

195. $\frac{dy}{dx} = \frac{p}{\sqrt{x^2+y^2-p^2}}$, $\frac{dp}{dx} = \frac{y}{\sqrt{x^2+y^2-p^2}}$.

196. $\frac{dy_1}{dx} = \frac{p_1}{2}$, $\frac{dp_1}{dx} = 0$, $\frac{dy_2}{dx} = \frac{p_2}{2}$, $\frac{dp_2}{dx} = 2y_2$.

197. $\frac{dy_1}{dx} = \frac{p_1}{2y_1}$, $\frac{dy_2}{dx} = \frac{p_2}{2y_2}$, $\frac{dp_1}{dx} = \frac{p_1^2}{4y_1^2}$, $\frac{dp_2}{dx} = \frac{p_2^2}{4y_2^2}$.

198. $\frac{dy_1}{dx} = -\frac{p_1}{2}$, $\frac{dy_2}{dx} = -\sqrt{p_2}$, $\frac{dp_1}{dx} = 2x$, $\frac{dp_2}{dx} = 0$;

$\frac{dy_1}{dx} = -\frac{p_1}{2}$, $\frac{dy_2}{dx} = \sqrt{p_2}$, $\frac{dp_1}{dx} = 2x$, $\frac{dp_2}{dx} = 0$.

199. $y^3 = C_1x^3 + C_2$. **200.** $y^3 = \ln^2 x$.

201. $x = C_1 \int \frac{dy}{\sqrt{G^2(y) - C_1^2}} + C_2$.

202. On the extremal $y = \frac{x^2 - x - 1}{2}$ is attained a strong minimum: $\min J = -\frac{5}{4}$. **203.** $p(x, y) = \frac{y^2 - x^2}{2xy}$. The extremals are semicircles $y = \sqrt{C_1^2 - (x - C_2)^2}$ with centres on the x-axis; $y = \sqrt{2C_1x - x^2}$ are extremals passing through the origin $O(0, 0)$; the field is the upper half-plane. **204.** An arc of the circle with centre at the point $O(0, 0)$ passing through the point $M_1(x_1, y_1)$ yields a strong minimum.

205. $xF\left(\frac{y}{x}\right) = C$.

206. The ellipses $3x^2 - 8xy + 6y^2 = C$. **207.** $x^3 + 2y^3 - 3xy^2 - 2x^2y = C$. **208.** $f = \sqrt{1 + y'^2}$. **209.** $f = xy\sqrt{y'}$.

210. $f = xyy'^2$. **211.** $f = \sqrt{\left(\frac{1}{x^2} + \frac{1}{y^2}\right)(x^2y'^2 + y^2)}$.

212. Catenary. **213.** *Hint.* The action integral $J = \int \sqrt{\frac{k}{\rho^4} + 2h} \times \sqrt{\rho^2 + \rho'^2}\, d\varphi$. **214.** The trajectories are the ellipses $\frac{x^2}{C} + \frac{2y^2}{2h - C} - \frac{2\cos\beta}{\sqrt{C(2h-C)}}\, xy = \frac{\sin^2\beta}{k}$.

215. The exact solution is $y = \frac{\sinh x}{\sinh 1} - x$.

216. Exact solutions, (a) $y \equiv 0$, (b) $y = x$.

217. The exact solution is $y = \frac{1}{2}(x^2 - x)$.

218. The exact solution is $y = \frac{2\sinh x}{\sinh 2} - x$. **219.** *Hint.* Seek the approximate solution in the form $y_n(x) = (1 - x^2)\sum_{k=0}^{n} \alpha_k x^{2k}$. The exact solution is $y = \cos\frac{\pi x}{2}$.

220. *Hint.* Take xy for the coordinate function. Then $z_1 = \frac{b^2 - a^2}{b^2 + a^2}\, xy$. **221.** *Hint.* For the coordinate functions take $\varphi_0(x, y) = x^2 + y^2$, $\varphi_1(x, y) = xy(1 - x - y)$, $\varphi_2(x, y) = x^2y(1 - x - y)$, ..., $\varphi_n(x, y) = x^n y(1 - x - y)$. Then $z_3(x, y) = x^2 + y^2 + xy(1 - x - y)[3.0401 - 0.0562(x + x^2)]$.

222. *Hint.* Seek the first approximation in the form $z_1(x, y) = \left(y^2 - \frac{x^2}{3}\right)\alpha(x)$. Then $z_1(x, y) = -\frac{3}{4}\left(1 - \frac{x}{6}\right) \times \left(y^2 - \frac{x^2}{3}\right)$.

223. $z_1(x, y) = \frac{3}{4}\left(y^2 - \frac{1}{3}x^2\right)\left(\frac{1 - 3^5}{1 - 3^6}x - \frac{2\cdot 3^5}{1 - 3^6}\, x^{-5} - 1\right)$.

224. $\lambda_n = 1 + n^2\pi^2$, $y_n(x) = \pm\sqrt{2}\sin n\pi x$ $(n = 1, 2, \ldots)$.

225. $\lambda_n = \frac{\ln^2 2 + 4n^2\pi^2}{4\ln^2 2}$, $y_n(x) = \pm\frac{\sin\left(\frac{n\pi}{\ln 2}\ln x\right)}{\sqrt{\ln\sqrt{2}\,\sqrt{x}}}$.

226. $\lambda_n = \frac{25 + 4n^2\pi^2}{4}$, $y_n(x) = \pm\frac{\sqrt{2}\sin(n\pi\ln x)}{\sqrt{x}}$ $(n = 1, 2, \ldots)$.

227. $\lambda_n = 1 - n^2$, $y_n(x) = \pm \sqrt{\frac{2}{\pi}} \sin nx$ $(n = 1, 2, \ldots)$.

228. $\lambda_n = -\frac{13 \ln^2 2 + 4n^2\pi^2}{4 \ln^2 2}$, $y_n(x) = \pm \frac{\sin\left[\frac{n\pi \ln(1+x)}{\ln 2}\right]}{\sqrt{\ln \sqrt{2}}\sqrt{1+x}}$

$(n = 1, 2, \ldots)$.

229. Take $y = 1 - x^2$ and obtain $\lambda_1 \leqslant \frac{35}{138}$. The exact value is $\lambda_1 = \frac{1}{4}$.

230. Take $y = x(1 - x)$, obtain $\lambda_1 \leqslant 10$. The exact value is $\lambda_1 = \pi^2$.

231. $\lambda_1^2 = 10$. The exact value is $\lambda_1^2 = \pi^2$.

232. $\lambda_1 = 0.493$. 233. $\lambda_1 = 6$, $z_1(x, y) = \alpha(x^2 + y^2 - 1)$.

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INDEX

absolute extremum 9, 51
action 173
action integral 173
additive inverse element 7
Akhiezer, N. I. 215
analytic form of the Jacobi condition 96
axiom
    identity 8
    symmetry 8
    triangle 8

belongs to a set 7
Bernstein, S. N. 215
Bernstein (theorem) 59
bilinear form 46
bilinear functional 46
bound
    greatest lower 7
    least upper 7
broken-line extremals 152

calculus of variations
    direct methods in 177ff
    elementary problems in 54
        generalization of 75
canonical form of Euler's equations 161
canonical system of Euler's equations 162
canonical variables 162
C-discriminant 93
central field 88
centre of a pencil of curves 88
circle (J-circle) 149
Clairaut's equation 170
Clairaut's theorem 136
Collatz, Lothar 215
condition(s)
    isoperimetric 126
    Jacobi's 98
    Jacobi's (analytic form of) 96
    Jacobi's strong 116
    Jacobi's sufficient 95
    Legendre's strong 100, 105, 116
    orthogonality 198
    strong (Legendre's) 100, 105, 116
    strong (Jacobi's) 116
    sufficient (of Legendre) 100, 105
    sufficient (for a minimum) 116
    sufficient (for a strong extremum) 102
    sufficient (for a weak extremum) 102
    sufficient (of Weierstrass) 102
    transversality 138
    Weierstrass 102
    Weierstrass-Erdman 152
conditional extremum 19, 118
conditional maximum 19
conditional minimum 19
conjugate point 94, 116, 117
continuity of a functional 33
continuous functional 33
coordinate functions, system of 180
corner point 154

Courant, R. 215
criterion, Sylvester's 12
critical force, Euler's 109
critical points 11
curve(s)
  distance between 31
  Lagrangian 55
  ε-neighbourhood of 33
  proximity of 27, 29

definite quadratic form 12
Demidovich, B. P. 215
Dido's problem 119, 122
differentiable functional 39
direct methods in the calculus of variations 177ff
Dirichlet integral 81
discontinuous functional 33
discontinuous problems 151, 156
discriminant (C-discriminant) 93
distance 8
  between curves 31
  geodesic 148
  (J-distance) 148
domain of definition of a functional 27

eigenfunctions 158, 188, 189
  normalized 189
eigenvalues 188
element
  additive inverse 7
  (belongs to a set) 7
  (is a member of a set) 7
  zero 7
elementary problem in calculus of variations, generalizations of 75
ellipse (J-ellipse) 149
Elsgolts, L. E. 6, 215
equation(s)
  Clairaut's 170
  of constraint 19
  Euler-Ostrogradsky 80
  Euler-Poisson 75
  Euler's (see Euler's equations) 54, 55
  Hamilton-Jacobi 165
epuation(s)
  Laplace 88
  Laplace (n-dimensional) 81
  Poisson 186
  Sturm-Liouville 188
  of surface of revolution 136
Erdman (see Weierstrass-Erdman conditions) 152
Euler-Ostrogradsky equation 80
Euler-Poisson equation 75
Euler-Poisson function 77
Euler's critical force, 109
Euler's equation(s) 54, 55
  canonical form of 161
  canonical system of 162
  Hamiltonian form of 161
  Hamiltonian system of 162
  integrability of 62
  invariance of 85ff
  system of 78
  Weierstrassian form of 73, 126
Euler's finite-difference method 177ff
Euler's problem 109
Euler's theorem 119
extrema (see extremum)
extremal(s) 55
  broken-line 152
  field of 88, 90
  included in a field 91
extremal field 90
extremum (extrema)
  absolute 9, 51
  conditional 19, 118
  of a function of many variables 9ff
  of functionals 27ff, 49
  necessary condition for 10, 49, 53
  relative 50
  strict (sufficient conditions for) 12
  strict conditional 21
  sufficient conditions for 102
  sufficient conditions for a strong 102
  sufficient conditions for a weak 102
extremum point 9

Fermat's principle 68
field
central 88
extremal 90
of extremals 88
proper 88
slope of 88
figuratrix 111
finite-difference method, Euler's 177
Fomin, S. V. 215
force, Euler's critical 109
form
bilinear 46
quadratic 12
formula, Green's 125
function(s)
coordinate (system of) 180
Euler-Poisson 77
Lagrangian 20
stationary 53
system of coordinate 180
Weierstrass 102, 111
functional(s) 27ff
bilinear 46
continuity of 33
continuous 33
dependent on functions of several independent variables 79
dependent on m functions 77
dependent on higher derivatives 75
differentiable 39
discontinuous 33
domain of definition of 27
extrema of 27ff, 49
increment of 38
linear 37
quadratic 46, 116
variation of 27ff, 38
second 46
second definition of 44, 45

gas flow, least resistance to 66
Gelfand, I. M. 215
geodesic(s) 133, 135
geodesic distance 148
geometry
Lobachevskian, Poincaré's model of 70
non-Euclidean 150
Goursat, É. 215
greatest lower bound 7
Green's formula 125
Gyunter, N. M. 215

Hamilton-Jacobi equation 165
Hamilton-Jacobi theory 161
Hamilton-Ostrogradsky principle 171, 173
Hamiltonian 161
Hamiltonian form of Euler's equations 161
Hamiltonian system of Euler's equations 162
Hilbert, D. 215
hyperbola (J-hyperbola) 149

identity axiom 8
increment of argument 29
increment of functional 38
indefinite quadratic form 12
inf 7
infimum 7
infinite points 69
integral
action 173
Dirichlet 81
intersection 7
invariance of Euler's equation 85
isoperimetric conditions 126
isoperimetric problem(s) 118, 126
reciprocity law of 122, 123

J-circle 149
J-distance 148
J-ellipse 149
J-hyperbola 149
J-length 148
J-line 148
Jacobi's condition 98
analytic form of 96
Jacobi's strong condition 116
Jacobi's sufficient condition 95
Jacobi's theorem 165

Kantorovich, L. V. 215
Kantorovich method 179, 185, 199
Kelvin's problem 125
Kiselev, A. I. 215
Krasnov, M. L. 215
Krylov, V. I. 215
Kudryavtsev, L. D. 215
Kuzmin, R. O. 215

Lagrangian curves 55
Lagrangian function 20
Lagrangian method of undetermined multipliers 20
Lagrangian multiplier method 20
Lagrangian problem 131
Laplace equation
  n-dimensional 81
  in polar coordinates 87, 88
Lavrentiev, M. A. 215
law
  reciprocity (of isoperimetric problems) 122, 123
  of refraction of light 159
least action, principle of
  in form of Jacobi 173
  in form of Lagrange 173
least resistance to a gas flow 66
least upper bound 7
Legendre, sufficient conditions 100, 105
Legendre's strong condition 100, 105, 116
length (J-length) 148
light, refraction of
  law of 159
  problem of 157
lines
  J-line 148
  parallel 69
linear functional 37
linear space 7
Liouville (see Sturm-Liouville boundary-value problem) 188
  (see Sturm-Liouville equation) 188
Lobachevskian geometry, Poincaré model of 70
Lyusternik, L. A. 215

Makarenko, G. I. 215
maximum
  conditional 19
  strict 50
  strict conditional 21
  strong relative 50
  weak relative 50
maximum point 9
mechanics, variational principles of 161, 171
member of a set 7
method(s)
  Euler's finite-difference 177ff
  Kantorovich 179, 185, 199
  Lagrangian (of undetermined multipliers) 20
  Lagrangian multiplier 20
  of reducing to ordinary differential equations 199
  Ritz 179
  of steepest descent 17
  variational (for finding eigenvalues and eigenfunctions) 188ff
metric space 8
Mikhlin, S. G. 216
minimizing sequence 180
minimum
  conditional 19
  strict conditional 21
  strong relative 50
  sufficient condition for 116
  weak relative 50
minimum point 10
Myškis, A. D. 216
moving boundary problems 137, 141

ε-neighbourhood of a curve (weak, strong) 33
non-Euclidean geometry 150
nonnegative quadratic form 11
norm 8
normalized eigenfunction 189
normed linear space 8

one-sided variations 151, 159
optical path length 68
orthogonality condition 198

Ostrogradsky (see Hamilton-Ostrogradsky principle) 171

parallel lines 69
Poincaré model of Lobachevskian geometry 70
point
  conjugate 94, 116, 117
  corner 154
  critical 11
  extremum 10
  infinite (point at infinity) 69
  maximum 10
  minimum 10
  stationary 11
  of strict maximum 9
  of strict minimum 9
Poisson equation 186
potential 172
principle(s)
  Fermat's 68
  Hamilton-Ostrogradsky 171, 173
  of least action in the form of Jacobi 173
  of least action in the form of Lagrange 173
  Rayleigh's 196, 199
  variational (of mechanics) 161, 171
problem(s)
  boundary-value, Sturm-Liouville 188
  Dido's 119, 122
  discontinuous 151, 156
  elementary (in calculus of variations) 54ff
  Euler's 109
  isoperimetric 118, 126
    reciprocity law of 122, 123
  Kelvin's 125
  Lagrangian 131
  moving boundary 137, 141
  of refraction of light 157
  Sturm-Liouville boundary-value 188
  variational (in parametric form) 72
proper field 88
proximity of curves 27, 29

quadratic form (definite, indefinite, nonnegative) 11, 12
quadratic functional 47, 116

Rayleigh's principle 196, 199
reciprocity law of isoperimetric problems 122, 123
refraction of light
  law of 159
  problem of 157
relative extrema 50
Ritz method 179
Rozhdestvensky, B. L. 216

sequence, minimizing 180
Shilov, G. E. 216
slope of field 88
Smirnov, V. I. 215, 216
space
  $C[a, b]$, $C_1[a, b]$, $C_n[a, b]$ 8
  linear 7
  metric 8
  normed linear 8
stationary functions 53
stationary points 11
steepest descent, method of 17
strict conditional extremum 21
strict conditional maximum 21
strict conditional minimum 21
strict extremum, sufficient conditions for 12
strict maximum, point of 9
strict minimum, point of 9
strong extrema 50
strong relative maximum 50
strong relative minimum 50
Sturm-Liouville boundary-value problem 188
Sturm-Liouville equation 188
subset 7
sufficient conditions for extremum 102
sufficient conditions of Legendre 100, 105
sufficient conditions for a minimum 116
sufficient conditions for a strong extremum 102
sufficient conditions for a weak extremum 102

sufficient conditions of Weierstrass 102
sup 7
supremum 7
surface of revolution, equation of 136
Sylvester's criterion for positive definiteness of a quadratic form 12
symmetry axiom 8
system of coordinate functions 180
system of Euler's equations 78

theorem
  Bernstein 59
  Clairaut's 136
  Euler's 119
  Jacobi's 165
  Weierstrass 9
theory, Hamilton-Jacobi 161ff
Tolstov, G. P. 216
transversality conditions 138
triangle axiom 8
  for norms 8
Tslaf, L. Ya. 6, 216

union 7

variables, canonical 162
variation(s) 29
  of a functional 27, 38
    second 46
    second definition of 44, 45
  one-sided 151, 159
variational methods for finding eigenvalues and eigenfunctions 157ff
variational principles of mechanics 161, 171
variational problems in parametric form 72

weak extrema 50
weak relative maximum 50
weak relative minimum 50
Weierstrass 51
  sufficient conditions of 102
Weierstrass condition 102
Weierstrass function 102, 111
Weierstrass theorem 9
Weierstrass-Erdman conditions 152
Weierstrass form of Euler's equation 73, 126

zero element 7

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